

A TOPOLOGICAL GRADING ON BORDERED HEEGAARD FLOER HOMOLOGY

VINICIUS GRIPP AND YANG HUANG

ABSTRACT. In a previous work [1], we defined an absolute grading on Heegaard Floer homology by homotopy classes of oriented 2-plane fields. In this paper, we generalize our construction to bordered Heegaard Floer homology.

1. INTRODUCTION

For a closed oriented 3-manifold Y , Ozsváth and Szabó [3] defined the Heegaard Floer homology groups $\widehat{HF}(Y)$, $HF^\infty(Y)$, $HF^-(Y)$ and $HF^+(Y)$, which are invariants of Y . These groups split into direct sums of groups by Spin^c structures, each of which is relatively graded. In [1], we constructed a canonical absolute grading for these groups taking values in the set of homotopy classes of oriented 2-plane fields, or equivalently, the set of homotopy classes of nonvanishing vector fields. In this paper, we will define a similar geometric grading on bordered Heegaard Floer homology.

We start by briefly reviewing the construction of bordered Heegaard Floer homology, following [2]. Consider a compact oriented 3-manifold Y with non-empty connected boundary. A parametrization of ∂Y is an orientation preserving diffeomorphism $\phi : \partial Y \rightarrow F$, where F is a closed oriented surface with a prescribed handle decomposition. According to [2], one can associate to F a differential graded algebra $\mathcal{A}(F)$. See § 2 for the precise definition of $\mathcal{A}(F)$. Then one defines the so-called *type A* and *type D* modules of Y , denoted by $\widehat{CFA}(Y)$ and $\widehat{CFD}(Y)$. The type *A* module $\widehat{CFA}(Y)$ is a right A^∞ -module over $\mathcal{A}(F)$. That means that there exist maps

$$m_l : \widehat{CFA}(Y) \otimes \mathcal{A}(F)^{\otimes(l-1)} \rightarrow \widehat{CFA}(Y),$$

satisfying the A^∞ -relations, see e.g. [2, Eq. (2.6)]. Here the tensor product is taken over an appropriate ring, as we will review in §3.3. The type *D* module $\widehat{CFD}(Y)$ is a left differential module over $\mathcal{A}(-F)$, that is there exists a map $\partial : \widehat{CFD}(Y) \rightarrow \widehat{CFD}(Y)$, which squares to 0 and which satisfies the Leibniz rule with respect to the left action of $\mathcal{A}(-F)$. It is also shown in [2] that if Y_1 and Y_2 are compact 3-manifolds such that $\partial Y_1 = -\partial Y_2$, then there is a homotopy equivalence

$$(1.0.1) \quad \Phi : \widehat{CFA}(Y_1) \widetilde{\otimes} \widehat{CFD}(Y_2) \rightarrow \widehat{CF}(Y_1 \cup_F Y_2).$$

Here $\widetilde{\otimes}$ denotes the derived tensor product. For a closed oriented 3-manifold Y , we denote by $\text{Vect}(Y)$ the set of homotopy classes of non-vanishing vector fields on Y . The goal of this paper is to prove the following theorems.

Theorem 1.1. *Given a parameterized surface F as above, there exist a groupoid $G(F)$, with a \mathbb{Z} -action denoted by λ^n for a given $n \in \mathbb{Z}$, and a grading function gr with values on $G(F)$ satisfying the following conditions:*

- (1) If a, b are two composable generators of $\mathcal{A}(F)$, then $\text{gr}(a \cdot b) = \text{gr}(a) \cdot \text{gr}(b)$.
- (2) If a is a generator of $\mathcal{A}(F)$, then $\text{gr}(\partial a) = \lambda^{-1} \text{gr}(a)$.

Remark 1.2. It turns out that $G(F)$ is by construction a set of co-oriented plane fields on $F \times [0, 1]$ modulo homotopy.¹ The multiplication rule is by the obvious stacking of plane fields when the boundary condition matches. Sometimes these plane fields can be realized as tight contact structures on $F \times [0, 1]$ with convex boundary, and $G(F)$ can be mapped into the (universal) contact category $\mathcal{C}(F)$ due to Honda [5]. But we shall not explore this issue any further in this paper.

Theorem 1.3. *For any compact 3-manifold Y with boundary F , there exist a set $S(Y)$ of non-vanishing vector fields on Y , admitting a right action by $G(F)$ and a left action by $G(-F)$, and a grading gr on $\widehat{CFA}(Y)$ and $\widehat{CFD}(Y)$ with values on $S(Y)$ such that*

- (a) *If x is a generator of $\widehat{CFA}(Y)$ and a_1, \dots, a_l are generators of $\mathcal{A}(F)$ such that $m_{l+1}(x; a_1, \dots, a_l) \neq 0$, then*

$$\text{gr}(m_{l+1}(x; a_1, \dots, a_l)) = \lambda^{l-1} \text{gr}(x) \cdot \text{gr}(a_1) \dots \text{gr}(a_l).$$

- (b) *If x is a generator of $\widehat{CFD}(Y)$, then $\text{gr}(\partial x) = \lambda^{-1} \text{gr}(x)$.*

Theorem 1.4. *Let Y_1 and Y_2 be compact 3-manifolds such that $\partial Y_1 = -\partial Y_2$. Then there exist a set $S(Y_1) \otimes S(Y_2)$ and a map $\Psi : S(Y_1) \otimes S(Y_2) \rightarrow \mathcal{P}(Y)$ such that*

$$\tilde{\text{gr}}(\Phi(a \otimes b)) = \Psi(\text{gr}(a) \otimes \text{gr}(b))$$

for any generators a in $\widehat{CFA}(Y_1)$ and b in $\widehat{CFD}(Y_2)$. Here $\tilde{\text{gr}}$ denotes the absolute grading in Heegaard Floer homology from [1].

The paper is organized as follows: In § 2, we first review the definition of the strand algebra $\mathcal{A}(F)$ associated to a parameterized closed surface F following [2]. Then we construct the groupoid $G(F)$ in which the grading on $\mathcal{A}(F)$ takes value, and give the proof for Theorem 1.1. We finish this section by comparing our geometric grading on $\mathcal{A}(F)$ with the previously constructed grading in [2]. § 3 is devoted to the construction of the “left- $G(-F)$ and right- $G(F)$ bimodule” $S(Y)$ in which the grading on $\widehat{CFA}(Y)$ and $\widehat{CFD}(Y)$ takes value. Some variations of the standard Pontryagin-Thom construction are made in this section which enable us to compute the relative gradings needed for the proof of Theorem 1.3. The proof of Theorem 1.4 is provided in § 4.

2. THE GRADING ON THE ALGEBRA

In this section, we construct the grading on the algebra $\mathcal{A}(\mathcal{Z})$. This grading takes values in a certain groupoid $G(\mathcal{Z})$. Before defining $G(\mathcal{Z})$ and the grading, we will quickly review the construction of $\mathcal{A}(\mathcal{Z})$. For a more thorough exposition, see [2].

¹By choosing a Riemannian metric on a 3-manifold, we can identify the set of nonvanishing vector fields with the set of co-oriented plane fields, modulo homotopy, by taking the orthogonal complement.

2.1. The construction of the algebra $\mathcal{A}(\mathcal{Z})$. The *strand algebra* $\mathcal{A}(\mathcal{Z})$ is defined as a subalgebra of $\mathcal{A}(4k)$. As a $\mathbb{Z}/2$ -vector space, $\mathcal{A}(4k)$ is generated by partial permutations (S, T, ϕ) , where S and T are subsets of $\{1, \dots, 4k\}$ containing the same number of elements and $\phi : S \rightarrow T$ is a bijection such that $\phi(i) \geq i$ for every $i \in S$. We can represent (S, T, ϕ) by a diagram with $4k$ points on the left and on the right and with strands connecting the set S on the left with the set T on the right. This diagram is required to have the smallest possible number of crossings. Each crossing corresponds to what is called an *inversion*, i.e. a pair of points $i, j \in \{1, \dots, 4k\}$ with $i < j$ and $\phi(i) > \phi(j)$. It follows from this definition that the strands either go up or stay horizontal if we read from left to right. The product of (S, T, ϕ) with (S', T', ϕ') is defined to be $(S, T', \phi' \circ \phi)$ provided that $T = S'$ and that the number of inversions of the diagram for $(S, T', \phi' \circ \phi)$ equals the sum of the number of inversions of the diagrams for (S, T, ϕ) with (S', T', ϕ') . Otherwise, the product is set to be 0. For each subset S , one can define an idempotent element $I(S) = (S, S, \mathbb{I}_S)$. One can also define a differential on $\mathcal{A}(4k)$ as follows. For a generator a of $\mathcal{A}(4k)$, let ∂a be the sum over all ways to smooth one crossing of a , where we require all the terms of this sum to have exactly one less intersection than a . In other words, if smoothing one crossings decreases the number of inversions by more than 1, we set that term to zero.

We denote by $[2k]$ the set $\{1, \dots, 2k\}$. A *pointed matched circle* \mathcal{Z} is a quadruple (Z, \mathbf{a}, M, z) consisting of an oriented circle Z , a set of $4k$ points \mathbf{a} in Z , a two-to-one function $M : \mathbf{a} \rightarrow [2k]$ and a basepoint $z \in Z \setminus \mathbf{a}$. We also require that 0-surgery on Z along the pairs of points that are matched by M yields to a single circle. A pointed matched circle gives rise to a surface $F(\mathcal{Z})$ of genus k , which we often denote by F . The surface F is obtained by starting with a disk whose oriented boundary is Z , attaching 1-handles along all the pairs matched by M and attaching a 0-handle to the boundary circle. We observe that we can find a self-indexing Morse function $f : F \rightarrow [0, 2]$ such that $Z = f^{-1}(3/2)$ and \mathbf{a} is the intersection between Z and the ascending manifolds from the index one critical points. We can identify $[2k]$ with the set of index one critical points $\{p_1, \dots, p_{2k}\}$.

By a *Reeb chord* ρ , we mean an oriented arc on $Z \setminus z$, with the same orientation as Z , whose boundary lies in \mathbf{a} . We denote by ρ^- the initial endpoint of ρ and by ρ^+ its final endpoint. We write $\rho = [\rho^-, \rho^+]$. A set $\boldsymbol{\rho} = \{\rho_1, \dots, \rho_m\}$ of Reeb chords is said to be *consistent* if both sets $\boldsymbol{\rho}^- := \{\rho_1^-, \dots, \rho_m^-\}$ and $\boldsymbol{\rho}^+ := \{\rho_1^+, \dots, \rho_m^+\}$ have exactly m elements. A consistent set of Reeb chords $\boldsymbol{\rho}$ gives rise to an element $a_0(\boldsymbol{\rho})$ in $\mathcal{A}(4k)$ given by

$$a_0(\boldsymbol{\rho}) = \sum_{\substack{S \subset \{0, \dots, 4k\} \\ S \cap (\boldsymbol{\rho}^- \cup \boldsymbol{\rho}^+) = \emptyset}} (S \cup \boldsymbol{\rho}^-, S \cup \boldsymbol{\rho}^+, \phi_S)$$

where $\phi_S|_S = \mathbb{I}$ and $\phi_S(\rho_i^-) = \rho_i^+$ for every i . Now, for every $\mathbf{s} \subset [2k]$, we can define the following idempotent

$$I(\mathbf{s}) := \sum_{\substack{S \subset \{0, \dots, 4k\} \\ M \text{ maps } S \text{ bijectively to } \mathbf{s}}} I(S).$$

We let $\mathcal{I}(\mathcal{Z})$ be the ring of idempotents, which is defined to be the algebra generated by the elements $I(\mathbf{s})$ for $\mathbf{s} \in [2k]$. The unit of this algebra is

$$\mathbf{I} := \sum_{\mathbf{s} \subset [2k]} I(\mathbf{s}).$$

We now define the algebra $\mathcal{A}(\mathcal{Z})$ to be the subalgebra of $\mathcal{A}(4k)$ generated by $\mathcal{I}(\mathcal{Z})$ and by the elements $a(\boldsymbol{\rho}) := \mathbf{I}a_0(\boldsymbol{\rho})\mathbf{I}$, for every consistent set of Reeb chords $\boldsymbol{\rho}$. The algebra $\mathcal{A}(\mathcal{Z})$ is generated as a $\mathbb{Z}/2$ -vector space by elements of the form $I(\mathbf{s})a(\boldsymbol{\rho})$. We note that if $I(\mathbf{s})a(\boldsymbol{\rho}) \neq 0$, then $M|_{\rho^-}$ and $M|_{\rho^+}$ are injective, $M(\boldsymbol{\rho}^-) \subset \mathbf{s}$ and $(\mathbf{s} \setminus M(\boldsymbol{\rho}^-)) \cap M(\boldsymbol{\rho}^+) = \emptyset$.

Recall the three different ways that two Reeb chords can intersect. A pair of Reeb chords $\{\rho_1, \rho_2\}$ is said to be *interleaved* if $\rho_i^- < \rho_j^- < \rho_i^+ < \rho_j^+$ for $i = j + 1$ or $i = j - 1$, and *nested* if $\rho_i^- < \rho_j^- < \rho_j^+ < \rho_i^+$ for $i = j + 1$ or $i = j - 1$. The Reeb chords ρ_1 and ρ_2 are said to *abut* if $\rho_1^+ = \rho_2^-$. In this case, one defines their *join* to be $\rho_1 \uplus \rho_2 := [\rho_1^-, \rho_2^+]$.

2.2. The groupoid $G(\mathcal{Z})$. Let $F = F(\mathcal{Z})$. We consider the bundle $TF \oplus \underline{\mathbb{R}} \rightarrow F$, where $\underline{\mathbb{R}}$ is the trivial real line bundle. We interpret this bundle as the pullback of the tangent bundle of a three-manifold in which F is embedded, so we call sections of this bundle vector fields on F . We will now construct a vector field $v'_0 : F \rightarrow TF \oplus \underline{\mathbb{R}}$. Let f be a self-indexing Morse function compatible with \mathcal{Z} as above. Consider its gradient vector field ∇f and modify it to first eliminate the index zero and index two critical points as follows. Let γ be the flow line passing through the basepoint z , which connects the index zero critical point to the index two critical point. Let $N(\gamma)$ denote a neighborhood of γ . Figure 1(a) illustrates ∇f restricted to $N(\gamma)$. We now define a nonvanishing vector field on $N(\gamma)$, which coincides with ∇f on $\partial N(\gamma)$, as shown in Figure 1(b). This picture determines the desired vector field up to homotopy relative to the boundary. This is similar to the construction in [1, §2]. Let v'_0 denote the vector field given by this construction in $N(\gamma)$ and by ∇f in the complement of $N(\gamma)$.

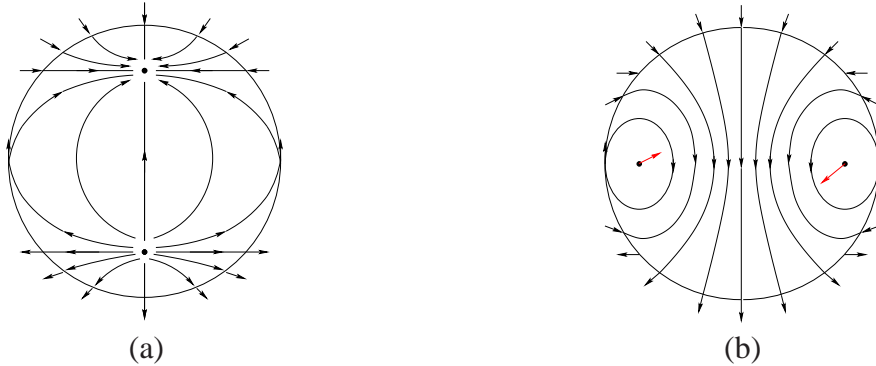


FIGURE 1. (a) The gradient vector field ∇f in a neighborhood of the flow line passing through z . (b) The non-vanishing vector field in the same neighborhood after modification. The red arrow on the left is pointing into the page and the arrow on the right is pointing out.

Note that each subset $\mathbf{s} \subset [2k]$ corresponds to a set of index one critical points of f , under the identification $[2k] = \{p_1, \dots, p_{2k}\}$. We denote by $\bar{\mathbf{s}}$ the subset $[2k] \setminus \mathbf{s}$. For $\mathbf{s} \subset [2k]$, let $\phi_{\mathbf{s}}$ be a bump function, which equals 1 at each point of \mathbf{s} and 0 outside of small neighborhoods of each point of \mathbf{s} . We denote by $|\mathbf{s}|$ the cardinality of the set \mathbf{s} .

Definition 2.1. For each $\mathbf{s} \in [2k]$, we define $v_{\mathbf{s}} : F \rightarrow TF \oplus \underline{\mathbb{R}}$ to be the vector field given by

$$v_{\mathbf{s}} = v'_0 + \phi_{\mathbf{s}} \frac{\partial}{\partial t} - \phi_{\bar{\mathbf{s}}} \frac{\partial}{\partial t}.$$

Here t denotes the \mathbb{R} -coordinate.

We can now define the grading set $G(\mathcal{Z})$.

Definition 2.2. For $\mathbf{s}, \mathbf{t} \in [2k]$, such that $|\mathbf{s}| = |\mathbf{t}|$, we define $G(\mathbf{s}, \mathbf{t})$ to be set of the homotopy class of nonvanishing vector fields on $F \times [0, 1]$ that restrict to $v_{\mathbf{s}}$ on $F \times \{0\}$ and to $v_{\mathbf{t}}$ in $F \times \{1\}$. We define $G(\mathcal{Z})$ to be the disjoint union of $G(\mathbf{s}, \mathbf{t})$ for all $\mathbf{s}, \mathbf{t} \subset [2k]$ such that $|\mathbf{s}| = |\mathbf{t}|$.

Given vector fields v, w on $F \times [0, 1]$ such that $v|_{F \times \{1\}} = w|_{F \times \{0\}}$, we can take their concatenation $v \cdot w$, which we see as a vector field on $F \times [0, 1]$. So given $[v] \in G(\mathbf{s}, \mathbf{t})$ and $[w] \in G(\mathbf{t}, \mathbf{u})$, we define their composition by $[v] \cdot [w] := [v \cdot w] \in G(\mathbf{s}, \mathbf{u})$. We now recall the definition of a groupoid.

Definition 2.3. A groupoid is the set of morphisms of a small category² in which every morphism is invertible.

We observe that $G(\mathcal{Z})$ is a groupoid, coming from a category whose objects are the vector fields $v_{\mathbf{s}}$ for $\mathbf{s} \subset [2k]$. The groupoid $G(\mathcal{Z})$ admits a \mathbb{Z} -action, defined as follows. We will denote the action of an integer $n \in \mathbb{Z}$ by λ^n . First observe that, since $\pi_3(S^2) \simeq \mathbb{Z}$, there is a \mathbb{Z} -action on the set of homotopy classes of nonvanishing vector fields on a ball B^3 relative to its boundary. Our sign convention³ is such that the Hopf map $S^3 \rightarrow S^2$ acts on B^3 by λ^{-1} . Let $[v] \in G(\mathcal{Z})$ and fix a ball B in the interior of $F \times [0, 1]$. For $n \in \mathbb{Z}$, we define $\lambda^n \cdot [v]$ to be the relative homotopy class of the vector field obtained by the acting on $v|_B$ by λ^n and keeping v constant outside B . We observe that

$$\lambda^n \cdot ([v] \cdot [w]) = (\lambda^n \cdot [v]) \cdot [w] = [v] \cdot (\lambda^n \cdot [w]).$$

2.3. A $G(\mathcal{Z})$ -grading on $\mathcal{A}(\mathcal{Z})$. Recall that the strand algebra $\mathcal{A}(\mathcal{Z})$ is generated as a $\mathbb{Z}/2$ vector field by all the elements of the form $I(\mathbf{s})a(\boldsymbol{\rho})$, where $\mathbf{s} \subset [2k]$ and $\boldsymbol{\rho} = \{\rho_1, \dots, \rho_m\}$ is a consistent set of Reeb chords. For every element $I(\mathbf{s})a(\boldsymbol{\rho}) \neq 0$, we will define its grading $\text{gr}(I(\mathbf{s})a(\boldsymbol{\rho})) \in G(\mathbf{s}, \mathbf{t})$, where $\mathbf{t} = M(\boldsymbol{\rho}^+) \cup (\mathbf{s} \setminus M(\boldsymbol{\rho}^-))$.

For a general $\mathbf{s} \subset [2k]$, in order to draw a picture of $v_{\mathbf{s}} : F \rightarrow TF \oplus \mathbb{R}$ away from the index 0 and 2 critical points, we will project it to a vector field on TF and decorate the zeros of this vector field using the following convention: an index one critical point p is decorated with “+” if $v_{\mathbf{s}} = \partial_t$ at p , and with “−” if $v_{\mathbf{s}} = -\partial_t$ at p .

We will define the grading function gr by steps as follows.

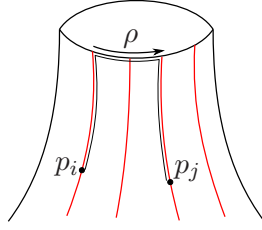
STEP 1. Assume that $\boldsymbol{\rho}$ consists of a single Reeb orbit ρ , such that $M(\rho^-) \neq M(\rho^+)$. We now construct $\text{gr}(I(\mathbf{s})a(\rho))$.

We will define a vector field $v_{(\mathbf{s}, \rho)}$ on $F \times [0, 1]$ such that $[v_{(\mathbf{s}, \rho)}] \in G(\mathbf{s}, \mathbf{t})$. Recall that we are identifying a point in $[2k]$ with its corresponding index one critical point. Let $p_i = M(\rho^-)$ and $p_j = M(\rho^+)$. So $\mathbf{t} = \{p_j\} \cup (\mathbf{s} \setminus \{p_i\})$. It follows from our construction in §2.2 that $v_{\mathbf{s}}$ and $v_{\mathbf{t}}$ only differ in small neighborhoods of p_i and p_j .

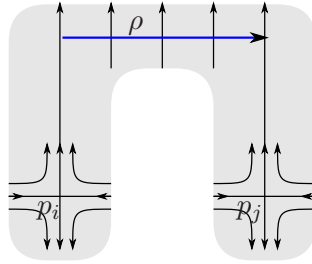
Let $\hat{\rho}$ be the arc from p_i to p_j consisting of three pieces: the gradient trajectory from p_i to ρ^- , the Reeb chord ρ and the gradient trajectory from p_j to ρ^+ , as shown in Figure 2.

²A small category is a category where the objects form a set.

³Note that our sign convention is the opposite of the usual one, but agrees with the one in [1].

FIGURE 2. Reeb chord ρ .

Let $N(\hat{\rho}) \subset F$ be a tubular neighborhood of $\hat{\rho}$. The vector field v_s restricted to $N(\hat{\rho})$ is depicted in Figure 3.

FIGURE 3. The neighborhood $N(\hat{\rho})$ of $\hat{\rho}$.

Define $v_{(s,\rho)}$ on $F \times \{0\}$ and $F \times \{1\}$ by setting it equal to v_s and v_t , respectively. Since $v_s = v_t$ on the complement of $N(\hat{\rho})$, we can extend $v_{(s,\rho)}$ on $(F \setminus N(\hat{\rho})) \times [0, 1]$, by requiring it to be invariant in the $[0, 1]$ -direction. The embedding $N(\hat{\rho}) \subset \mathbb{R}^2$, as shown in Figure 3, gives rise to a trivialization of $TF|_{N(\hat{\rho})}$ and, therefore, we obtain a trivialization of $T(F \times [0, 1])|_{N(\hat{\rho}) \times [0, 1]}$. We observe that, under the identification given by this trivialization, $v_s|_{N(\hat{\rho})}^{-1}(0, 0, 1) = p_i$ and $v_t|_{N(\hat{\rho})}^{-1}(0, 0, 1) = p_j$. The points p_i and p_j are framed codimension two submanifolds of $N(\hat{\rho})$. By the relative Pontryagin-Thom construction, in order to define a nonvanishing vector field on $N(\hat{\rho}) \times [0, 1]$ with the given boundary condition, it is enough to choose a framed 1-manifold, whose intersection with the boundary is $\{p_i\} \times \{0\} \cup \{p_j\} \times \{1\}$ with the given framing. We choose a framed 1-manifold as follows. Let $\gamma : [0, 1] \rightarrow F$ be a smoothing of $\hat{\rho}$ such that $\gamma(0) = p_i$ and $\gamma(1) = p_j$. Let $\tilde{\gamma} : [0, 1] \rightarrow F \times [0, 1]$ be the arc defined by $\tilde{\gamma}(t) = (\gamma(t), t)$. Since $F \times \{t\}$ is always transverse to $\tilde{\gamma}$, the embedding $N(\hat{\rho}) \subset \mathbb{R}^2$ gives a canonical framing on $\tilde{\gamma}$. Now, using this framed 1-manifold, the Pontryagin-Thom construction allows us to extend $v_{(s,\rho)}$ to the interior of $N(\hat{\rho}) \times [0, 1]$. We note that $v_{(s,\rho)}|_{N(\hat{\rho}) \times [0, 1]}$ is well-defined up to homotopy relative to the boundary. We now define $\text{gr}(I(s)a(\rho))$ to be the homotopy class of $v_{(s,\rho)}$, which is an element of $G(s, t)$.

It will be useful later to have a more concrete description of $\text{gr}(I(s)a(\rho))$. To do so, we view a vector field on $F \times [0, 1]$ as a smooth one-parameter family of nonvanishing sections $F \rightarrow TF \times \mathbb{R}$, indexed by $t \in [0, 1]$. We will, in fact, define a family of such sections $\{v_{(s,\rho)}^t\}_{t \in [0, 1]}$. This family can be explicitly defined by a composition of three bifurcations and necessary isotopies, which we now describe.

Consider the following model situation: Let Ξ^0 be a singular vector field on the unit disk $D \subset \mathbb{R}^2$ with two saddle points p, q as depicted in Figure 4(a). Then there exists a 1-parameter family of vector fields Ξ^t , for $0 \leq t \leq 1$, such that

- each Ξ^t has only two saddle points which are p and q , and Ξ^t is constant near ∂D as t goes from 0 to 1,
- for exactly one t , say $t = 1/2$, the vector field Ξ^t has a saddle-saddle connection from q to p .

See Figure 4 for a pictorial illustration of Ξ^t . We call the one-parameter family $\{\Xi^t\}_{t \in [0,1]}$, a *bifurcation*. Notice that in the situation of Figure 4, we decided to fix the unstable trajectories of p and the stable trajectories of q throughout the homotopy, however, we could instead fix the stable trajectories of p and the unstable trajectories of q throughout the homotopy to define another similar one-parameter family of vector fields with the same boundary condition, which we also call a bifurcation.

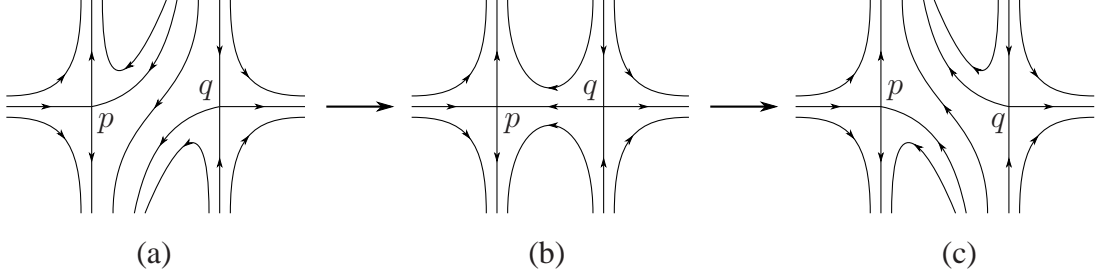


FIGURE 4. A bifurcation.

We can now define $v_{(s,\rho)}^t$ to be constant and equal to v_s in the complement of $N(\hat{\rho}) \times [0, 1]$. In $N(\hat{\rho}) \times [0, 1]$, we define $v_{(s,\rho)}^t$ via a composition of bifurcations and isotopies, as shown in Figure 5.

The family $\{v_{(s,\rho)}^t\}$ gives rise to a vector field on $F \times [0, 1]$, which we still denote $\{v_{(s,\rho)}^t\}$. As before, the embedding $N(\hat{\rho}) \subset \mathbb{R}^2$, as in Figure 3, induces a trivialization of $T(N(\hat{\rho}) \times [0, 1])$. We observe that, in $N(\hat{\rho}) \times [0, 1]$, the framed arc $(\{v_{(s,\rho)}^t\})^{-1}(0, 0, 1)$ is isotopic, and hence cobordant, to the arc $\tilde{\gamma}$. Moreover their framings coincide under the isotopy. Therefore, by the Pontryagin-Thom construction, this $\{v_{(s,\rho)}^t\}$ is a representative of homotopy class $\text{gr}(I(s)a(\rho))$.

STEP 2. Now assume that ρ still consists of only one Reeb orbit ρ , but $M(\rho^-) = M(\rho^+)$.

Let $p = M(\rho^-) = M(\rho^+)$ and let $\hat{\rho}'$ be the union of ρ and the flow lines connecting p to $\rho^- \cup \rho^+$. We construct a one-parameter family $\{\Theta^t\}_{t \in [0,1]}$ of vector fields on F as follows. Set $\Theta^0 = v_s$ and $\Theta^t \equiv \Theta^0$, for $t \in [0, 1]$, outside $N(\hat{\rho}')$. Fix a small $\varepsilon > 0$. For $t \in [0, \varepsilon]$, define Θ^t in $N(\hat{\rho}')$ to be the homotopy which creates an extra pair of singular points near p decorated with negative signs, along the unstable trajectories of p as depicted in Figure 6. More precisely, under the projection to TF , we create a pair of canceling critical points μ of index one and ν of index two, lying on the flow line of ∇f connecting p to ρ^+ . Consider the (broken) arc $\hat{\rho}$ from p to ν , which is the union of the trajectory from p to ρ^- , ρ and the trajectory from ν to ρ^+ . Now we can repeat the method from Step 1 for $\Theta^\varepsilon|_{N(\hat{\rho})}$ and obtain a homotopy $\Theta^t|_{N(\hat{\rho})}$ for $t \in [\varepsilon, 1 - \varepsilon]$, which exchanges the signs of the index one critical points. We define Θ^t in $N(\hat{\rho}') \setminus N(\hat{\rho})$ to be constant and equal to Θ^ε . For $t \in [1 - \varepsilon, 1]$, let $\Theta^t|_{N(\hat{\rho})}$ be the homotopy which cancels the extra pair of “negative” singular

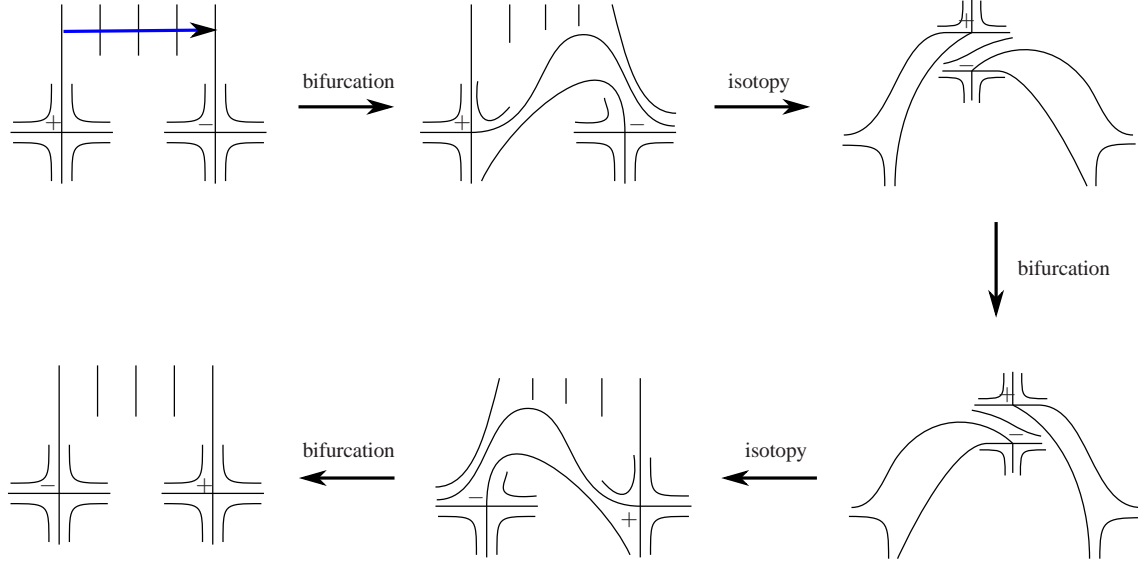


FIGURE 5. A sequence of three bifurcations which defines the grading of ρ . Arrows are omitted for simplicity of the picture (cf. Figure 3).

points. The family $\{\Theta^t\}_{t \in [0,1]}$ gives rise to a vector field, which is again denoted by $v_{(s,\rho)}$. Finally $\text{gr}(I(s)a(\rho))$ is defined to be the homotopy class of $v_{(s,\rho)}$.

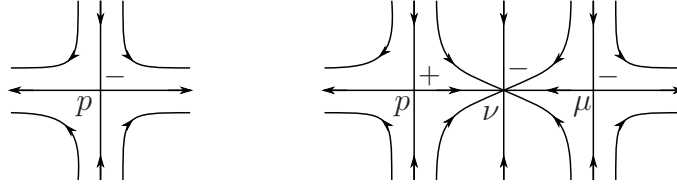


FIGURE 6. Creating a canceling pair of critical points with negative sign.

STEP 3. The general case.

Suppose $\rho = \{\rho_1, \dots, \rho_l\}$. Note that the choice of a basepoint z and an orientation on Z induce an ordering on \mathbf{a} : if we start from z and follow the positive orientation on Z , then $a_i < a_j$ if and only if we meet a_i before a_j , where $a_i, a_j \in \mathbf{a}$. We define an ordering on ρ by setting $\rho_i < \rho_j$ whenever $\rho_i^+ > \rho_j^+$ in \mathbf{a} . Up to re-ordering, we may assume that $\rho_1 < \rho_2 < \dots < \rho_l$. We want to define a relative homotopy class $\text{gr}(I(s)a(\rho)) \in G(v_s, v_t)$, where $\mathbf{t} = M(\rho^+) \cup (s \setminus M(\rho^-))$. First, for every point in $(M(\rho^-) \cap M(\rho^+)) \setminus (\rho^- \cap \rho^+)$, we create a pair of canceling “negative” singular points, as follows. If $p = M(\rho_i^+) = M(\rho_j^-)$, then we create a pair of “negative” singular points on the flow line connecting p to ρ_i^+ , as in Step 2. This construction gives rise to a vector field v_ε in $F \times [0, 1]$ similar to $\{\Theta|_t\}_{t \in [0, \varepsilon]}$ from Step 2. We also consider the vector field $v_{-\varepsilon}$, which corresponds to canceling the “negative” singular points added to v_t . Now consider the arcs $\hat{\rho}_i$ associated to ρ_i as before, namely, $\hat{\rho}_i$ is the union of ρ_i with the gradient trajectories connecting index one critical points to $\rho_1^- \cup \rho_1^+$. Note that $\hat{\rho}_i$ always connects a “positive” saddle to a “negative” saddle. Now we can define $v_{(s,\rho)}^1$ to be the homotopy class of the vector field supported on $N(\hat{\rho}_1)$

defined in Step 1. We repeat the same procedure for ρ_2, \dots, ρ_l , such that for every $i \geq 2$, the vector field $v_{(s,\rho)}^i$ corresponding to ρ_i is supported in $N(\hat{\rho}_i)$. In particular,

$$v_{(s,\rho)}^{i-1}|_{F \times \{1\}} = v_{(s,\rho)}^i|_{F \times \{0\}}.$$

Let $v_{(s,\rho)}$ be the concatenation

$$(2.3.1) \quad v_{(s,\rho)} := v_\varepsilon \cdot v_{(s,\rho)}^1 \cdot \dots \cdot v_{(s,\rho)}^l \cdot v_{-\varepsilon}.$$

Finally, we define $\text{gr}(I(s)a(\rho))$ to be the relative homotopy class of $v_{(s,\rho)}$, which is an element of $G(s, t)$.

2.4. The properties of the grading on $\mathcal{A}(\mathcal{Z})$. We now show that the grading we constructed in the previous subsection satisfies the desired properties.

Proposition 2.4. *The grading function $\text{gr} : \mathcal{A}(\mathcal{Z}) \rightarrow \mathcal{G}(\mathcal{Z})$ constructed above defines a grading on the dg algebra $\mathcal{A}(\mathcal{Z})$, i.e., it satisfies the following:*

- For any two sets of Reeb chords ρ, σ , if $I(s)a(\rho)I(t)a(\sigma) \neq 0$, then

$$\text{gr}(I(s)a(\rho)) \cdot \text{gr}(I(t)a(\sigma)) = \text{gr}(I(s)a(\rho)I(t)a(\sigma)),$$

- For any ρ , if $\partial(I(s)a(\rho)) \neq 0$, then

$$\text{gr}(\partial(I(s)a(\rho))) = \lambda^{-1} \cdot \text{gr}(I(s)a(\rho)).$$

Proof. For each index one critical point $p_i \in [2k]$, we denote by h_i the core of the corresponding 1-handle and by σ_i the Reeb chord connecting the two points in $h_i \cap Z$. We also denote by $N(z)$ and $N(p_i)$ small neighborhoods of z and p_i . Let \mathfrak{N} be a small neighborhood of

$$(Z \setminus N(z)) \cup \bigcup_{i=1}^{2k} (h_i \setminus N(p_i)).$$

We can choose an orientation-preserving embedding $\mathfrak{N} \hookrightarrow \mathbb{R}^2$ such that $Z \cap \mathfrak{N}$ is parallel to the horizontal vector ∂_x and its orientation is positive with respect to ∂_x , and such that $h_i \cap \mathfrak{N}$ is parallel to the vertical vector ∂_y , see Figure 7(a). Let $\tilde{\mathfrak{N}} := \mathfrak{N} \cup \bigcup_i N(p_i)$. So we can extend the embedding from above to an immersion $\tilde{\mathfrak{N}} \hookrightarrow \mathbb{R}^2$ such that $h_i \cap N(p_i)$ maps to a half-circle, see Figure 7(b). This immersion induces a trivialization of $TF|_{\tilde{\mathfrak{N}}}$. We obtain a trivialization of $(TF \oplus \mathbb{R})|_{\tilde{\mathfrak{N}}}$, which induces a $[0, 1]$ -invariant trivialization τ of $T(F \times [0, 1])$ over $\tilde{\mathfrak{N}}$.



FIGURE 7

Fix a generator $I(s)a(\rho) \in \mathcal{A}(\mathcal{Z})$. We write $\rho = \{\rho_1, \dots, \rho_l\}$ and we order the Reeb chords as in Step 3 above. We now define a one-manifold $Q_{(s,\rho)}$ in $\tilde{\mathfrak{N}} \times [0, 1]$. Let $\gamma_i \subset \tilde{\mathfrak{N}}$ be a smoothing of the union⁴ of ρ_i with the gradient flow trajectories connecting $M(\rho_i^-) \cup M(\rho_i^+)$ to $\rho_i^- \cup \rho_i^+$. We

⁴If $M(\rho_i^+) \neq M(\rho_j^-)$ for every j or $\rho_i^+ = \rho_j^-$ for some j , then this union is just $\hat{\rho}_i$. Otherwise, we need to add a small segment connecting $M(\rho_i^+)$ to the “negative” index one critical point corresponding to ρ_i^+ .

can perturb the arcs γ_i on F such that the interior of every two of these arcs intersect, at most, at one point. This happens precisely for an interleaved pair of Reeb chords. We define arcs $\tilde{\gamma}_i$ on $F \times [0, 1]$ by $\tilde{\gamma}_i(t) = (\gamma_i(t), t)$. Now if $\gamma_i(t) = \gamma_j(t)$ for some $t \in (0, 1)$, for $i < j$, we perturb $\tilde{\gamma}_i$ near t so that $\tilde{\gamma}_i(t) < \tilde{\gamma}_j(t)$. Hence the arcs $\tilde{\gamma}_i$ are all pairwise disjoint. So we can define $Q_{(s, \rho)}$ to be the union of $\tilde{\gamma}_i$ for all i and the constant arcs $p \times [0, 1]$, where $p \in s \setminus M(\rho^-)$. We observe that the $Q_{(s, \rho)} \cap \mathfrak{N}$ can be represented by the strand diagram corresponding to ρ . Here if ρ_i and ρ_j intersect and $i < j$, then ρ_i goes under ρ_j . See Figure 8 for examples of $Q_{(s, \rho)} \cap \mathfrak{N}$.

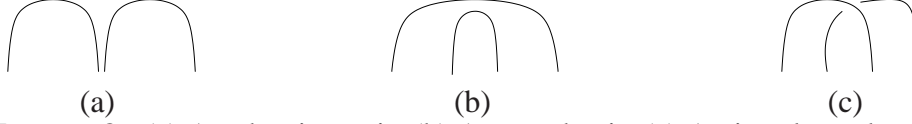


FIGURE 8. (a) An abutting pair. (b) A nested pair. (c) An interleaved pair.

We shall use the Pontryagin-Thom construction to prove both assertions of the proposition. For the first one, let $I(s)a(\rho)$ and $I(t)a(\sigma)$ be generators of $\mathcal{A}(F)$ whose product is nonzero. Recall that the *join* $\rho \uplus \sigma$ is obtained from the union $\rho \cup \sigma$ where for every abutting pair (ρ, σ) with $\rho \in \rho$ and $\sigma \in \sigma$ is substituted by $\rho \uplus \sigma$. It follows that if $I(s)a(\rho)I(t)a(\sigma) \neq 0$, then

$$(2.4.1) \quad I(s)a(\rho)I(t)a(\sigma) = I(s)a(\rho)a(\sigma) = I(s)a(\rho \uplus \sigma).$$

Let $v_{(s, \rho)}$, $v_{(t, \sigma)}$ and $v_{(s, \rho \uplus \sigma)}$ be the vector fields as in the construction of the grading, whose homotopy classes are $\text{gr}(I(s)a(\rho))$, $\text{gr}(I(t)a(\sigma))$ and $\text{gr}(I(s)a(\rho \uplus \sigma))$, respectively. We want to show that the product $v_{(s, \rho)} \cdot v_{(t, \sigma)}$ is homotopic to $v_{(s, \rho \uplus \sigma)}$. We clearly only need to look at the restriction of these vector fields to $\bar{\mathfrak{N}} \times [0, 1]$. Using the trivialization τ we can see the restriction of each of these vector fields as maps $\bar{\mathfrak{N}} \rightarrow S^2$. Up to a small perturbation of the vector fields, we can assume that $\mathbf{n} = (0, 0, 1) \in S^2$ is a regular value of all of these three maps. Then,

$$\begin{aligned} v_{(s, \rho)}^{-1}(\mathbf{n}) &\simeq Q_{(s, \rho)}, \\ v_{(t, \sigma)}^{-1}(\mathbf{n}) &\simeq Q_{(t, \sigma)}, \\ v_{(s, \rho \uplus \sigma)}^{-1}(\mathbf{n}) &\simeq Q_{(s, \rho \uplus \sigma)}. \end{aligned}$$

Here the symbol \simeq denotes relative framed cobordism. The framing on each of these one-manifolds is trivial in $\mathfrak{N} \times [0, 1]$ and has a standard form near every p_i ⁵. We now want to concatenate $Q_{(s, \rho)}$ and $Q_{(t, \sigma)}$.

Write $\rho = \{\rho_1, \dots, \rho_l\}$ and $\sigma = \{\sigma_1, \dots, \sigma_m\}$ with the ordering given as in Step 3 of the construction of the grading. Observe that $Q_{(s, \rho)}$ is isotopic to the concatenation of one-manifolds $Q_{1,1} \cdot \dots \cdot Q_{1,l}$ in $F \times [0, l]$ where each $Q_{1,i}$ consists of the union of the arc corresponding to ρ_i and constant arcs. Note that the one-manifolds $Q_{1,i}$ all have the same number of arcs. Similarly, we can write $Q_{(t, \sigma)}$ as a concatenation of one-manifolds $Q_{2,1} \cdot \dots \cdot Q_{2,m}$ in $F \times [0, m]$ corresponding to $\sigma_1, \dots, \sigma_m$. So

$$Q_{(s, \rho)} \cdot Q_{(t, \sigma)} \cong Q_{1,1} \cdot \dots \cdot Q_{1,l} \cdot Q_{2,1} \cdot \dots \cdot Q_{2,m}.$$

Here \cdot denotes concatenation and \cong denotes isotopy relative to the boundary, where we identify $F \times [0, 1] \cong (F \times [0, l]) \cdot (F \times [0, m])$. Now we want to reorder this concatenation in order to

⁵The standard framing is a rotation by π in $N(p_i)$ either positively or negatively depending on whether $p_i = M(\rho^-)$ or $p_i = M(\rho^+)$ for the corresponding Reeb chord ρ , but it does not depend on anything else.

obtain the decomposition of $Q_{(s, \rho \uplus \sigma)}$ defined as above. We will move $Q_{2,1}, \dots, Q_{2,m}$ to the left one by one, as necessary, which we explain in what follows. We start with $Q_{2,1}$. First note that if $\rho_i \cap \sigma_j = \emptyset$, then $Q_{1,i} \cdot Q_{2,j} \cong Q_{2,j} \cdot Q_{1,i}$. So we can move $Q_{2,1}$ to the left of $Q_{1,i}$, whenever $\rho_i^+ < \sigma_1^-$. If there exists ρ_i such that ρ_i and σ_1 abut then we stop at $Q_{1,i} \cdot Q_{2,1}$, since this concatenation is isotopic to the one-manifold corresponding to $\rho_i \uplus \sigma_1$ in the decomposition of $Q_{(s, \rho \uplus \sigma)}$. If that is the case, we can move $Q_{1,i} \cdot Q_{2,1}$ to the left of all the terms $Q_{1,i'}$ for which $\rho_{i'}^+ < \sigma_1^+$. In fact, let $Q_{1,i'}$ be such a term, i.e. $\rho_i^+ < \rho_{i'}^+ < \sigma_1$. Then $\{\rho_i \uplus \sigma_1, \rho_{i'}\}$ has to be nested, otherwise $a(\rho)a(\sigma) = 0$. So $Q_{1,i'} \cdot (Q_{1,i} \cdot Q_{2,1}) \cong (Q_{1,i} \cdot Q_{2,1}) \cdot Q_{1,i'}$. Now assume that there does not exist ρ_i such that ρ_i and σ_1 abut. Let ρ_i be such that $\rho_i \cap \sigma \neq \emptyset$. Then $\{\rho_i, \sigma_1\}$ has to be nested, otherwise $a(\rho)a(\sigma) = 0$. Therefore $Q_{1,i} \cdot Q_{2,1} \cong Q_{2,1} \cdot Q_{1,i}$. We now proceed analogously with $Q_{2,2}, \dots, Q_{2,m}$. After all these isotopies, we obtain the decomposition of $Q_{(s, \rho \uplus \sigma)}$ by one-manifolds corresponding to the Reeb chords in $\rho \uplus \sigma$. Therefore we conclude that

$$Q_{1,1} \cdot \dots \cdot Q_{1,l} \cdot Q_{2,1} \cdot \dots \cdot Q_{2,m} \cong Q_{(s, \rho \uplus \sigma)}.$$

Note that the framing on the one-manifold corresponding to an abutting pair $\{\rho_i, \sigma_j\}$ is the same as the framing on $Q_{1,i} \cdot Q_{2,j}$. Therefore, since the framings on all of these manifolds are standard, it follows that all these isotopies give rise to framed cobordisms. Therefore

$$\text{gr}(I(s)a(\rho))\text{gr}(I(t)a(\sigma)) = \text{gr}(I(s)a(\rho \uplus \sigma)).$$

The first assertion follows from (2.4.1).

Now let $I(s)a(\rho)$ be a generator of $\mathcal{A}(F)$ and let $v_{(s, \rho)}$ and $Q_{(s, \rho)}$ be as above. Recall that the differential of $I(s)a(\rho)$ is given by resolving one crossing of $I(s)a(\rho)$ at a time. We denote by ρ_j the sets of Reeb chords, such that $\partial(I(s)a(\rho)) = \sum_j I(s)a(\rho_j)$, and let $v_{(s, \rho_j)}$ and $Q_{(s, \rho_j)}$ be as above. So, resolving a crossing of the projection of $Q_{(s, \rho)}$ onto F is equivalent to doing a 0-surgery on $Q_{(s, \rho)}$, leading to a one-manifold, which is isotopic to $Q_{(s, \rho_j)}$ for some j . We observe that the framing on the result of the 0-surgery is one unit less⁶ than the framing on $Q_{(s, \rho)}$. Therefore

$$\text{gr}(I(s)a(\rho_j)) = \lambda^{-1} \text{gr}(I(s)a(\rho)).$$

Hence the differential decreases the grading by 1. □

2.5. Comparison with the grading by a non-commutative group. We now compare our topological grading constructed in §2.3 with the gradings on $\mathcal{A}(\mathcal{Z})$ defined in [2].

We first recall the definition of the non-commutative groups in which the gradings defined in [2] takes values. The group $G'(4k)$ is a \mathbb{Z} -central extension of $H_1(Z \setminus z, \mathbf{a})$. In order to give a more concrete definition of $G'(4k)$, we need to recall a few definitions from [2]. For a Reeb chord α in $Z \setminus z$ and $p \in Z \setminus z$, let $m(p, \alpha)$ be the average multiplicity with which α covers p , i.e. $m(p, \alpha) = 1/2$ for a boundary point, $m(p, \alpha) = 1$ for an interior point and $m(p, \alpha) = 0$, otherwise. One can extend m bilinearly to a function $m : H_1(Z \setminus z, \mathbf{a}) \times H_0(\mathbf{a}) \rightarrow \frac{1}{2}\mathbb{Z}$. For $\alpha_1, \alpha_2 \in H_1(Z \setminus z, \mathbf{a})$, one can define $L(\alpha_1, \alpha_2) = m(\partial\alpha_1, \alpha_2)$, where $\partial : H_1(Z \setminus z, \mathbf{a}) \rightarrow H_0(\mathbf{a})$ is the boundary map. Also, for $\alpha \in H_1(Z \setminus z, \mathbf{a})$, let $\varepsilon(\alpha) \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})$ be $\frac{1}{4}$ times the number of parity

⁶Note that, using our sign conversions, the writhe of $Q_{(s, \rho)}$ is one unit more than the writhe of $Q_{(s, \rho_j)}$ with respect to the canonical projection to F .

changes in $\alpha \bmod 1$. One can now define $G'(4k) = \{(j, \alpha) \in \frac{1}{2}\mathbb{Z} \times H_1(Z \setminus z, \mathbf{a}) \mid \varepsilon(\alpha) = j \bmod 1\}$. The multiplication is defined by

$$(j_1, \alpha_1) \cdot (j_2, \alpha_2) = (j_1 + j_2 + L(\alpha_1, \alpha_2), \alpha_1 + \alpha_2).$$

It follows from [2, Prop. 3.37] that this operation defines a multiplication in $G'(4k)$. For an element $g = (j, \alpha) \in G'(4k)$, the number $j \in \frac{1}{2}\mathbb{Z}$ is called the *Maslov component* of g and α is called the *Spin^c component* of g .

Given an element $a \in \mathcal{A}(4k)$, it determines a class $[a] \in H_1(Z \setminus z, \mathbf{a})$. We denote by $\text{inv}(a)$ the number of inversions of a . Write $a = (S, T, \phi)$. Let $\iota(a) = \text{inv}(a) - m(S, [a])$. Then one can define

$$\text{gr}'(a) = (\iota(a), [a]).$$

It follows from [2, Prop. 3.39] that $\text{gr}'(a) \in G'(4k)$. Moreover gr' is invariant under adding horizontal strands. Let $\rho = \{\rho_1, \dots, \rho_n\}$ be a set of Reeb chords and $\mathbf{s} \subset [0, 2k]$ be such that $I(\mathbf{s})a(\rho) \neq 0$. We can see ρ as an element of $\mathcal{A}(4k)$ with no horizontal strands. So $\text{gr}'(I(\mathbf{s})a(\rho)) = \text{gr}'(\rho)$. Let $|\rho|$ denote the number of elements of ρ , and let $|\text{ab}(\rho)|$ and $|\text{int}(\rho)|$ denote the number of abutting and interleaved pairs in ρ , respectively. Then a calculation shows that

$$(2.5.1) \quad \iota(\rho) = -\frac{|\rho|}{2} - \frac{|\text{ab}(\rho)|}{2} - |\text{int}(\rho)|.$$

By making some non-canonical choices, one can also define a *refined grading* taking values on a subgroup of $G'(4k)$, see [2, §3.3.2]. That is necessary for the gluing theorems to behave well with respect to grading. Alternatively, as suggested in [2, Rem. 10.44], one could consider a more canonical subset of $G'(4k)$, as follows. Let $M_* : H_0(\mathbf{a}) \rightarrow H_0([2k])$ denote the pushforward of the map $M : \mathbf{a} \rightarrow [2k]$. Define $G'(\mathcal{Z})$ to be the set of elements (j, α) in $G'(4k)$ such that $M_*(\partial\alpha) = \mathbf{t} - \mathbf{s}$, for $\mathbf{t}, \mathbf{s} \subset [2k]$, with $|\mathbf{t}| = |\mathbf{s}|$. We observe that $G'(\mathcal{Z})$ is a groupoid and that $\text{gr}'(a) \in G'(\mathcal{Z})$ for every homogeneous element $a \in \mathcal{A}(\mathcal{Z})$. We now have the following proposition.

Proposition 2.5. *There exists a homomorphism $\mathcal{F} : G(\mathcal{Z}) \rightarrow G'(\mathcal{Z})$ such that $\mathcal{F}(\text{gr}(a)) = \text{gr}'(a)$ for every homogeneous element $a \in \mathcal{A}(\mathcal{Z})$.*

Proof. Let \mathfrak{N} and $\bar{\mathfrak{N}}$ be as in the proof of Proposition 2.4. Let τ be the trivialization of $T(F \times [0, 1])$ in $\mathfrak{N} \times [0, 1]$ constructed in that proof. We extend this trivialization to a trivialization of $T(F \times [0, 1])$ arbitrarily.

For each $\mathbf{s} \subset [2k]$, we now see $v_{\mathbf{s}}$ as a map $F \rightarrow S^2$. We can slightly perturb the vector fields $v_{\mathbf{s}}$ so that $(0, 0, 1)$ is a regular value of these maps. Observe that $v_{\mathbf{s}}^{-1}(0, 0, 1) = \mathbf{s} \cup P$, where P is a set of points in the complement of $\bar{\mathfrak{N}}$ and does not depend on \mathbf{s} .

Now let $[v] \in G(\mathcal{Z})$. Then $[v] \in G(\mathbf{s}, \mathbf{t})$, for some $\mathbf{s}, \mathbf{t} \subset [2k]$, such that $|\mathbf{s}| = |\mathbf{t}|$. We see the vector field v as a map $F \times [0, 1] \rightarrow S^2$. We can slightly homotope v in $F \times (0, 1)$ so that $(0, 0, 1)$ is a regular value of v . Now consider $L_v := v^{-1}(0, 0, 1)$. Observe that $L_v \cap (F \times \{0\}) = (\mathbf{s} \times \{0\}) \cup P$ and $L_v \cap (F \times \{1\}) = (\mathbf{t} \times \{1\}) \cup P$. Since $H_1(F)$ is generated by $h_i \cup \sigma_i \subset \mathfrak{N} \times [0, 1]$ for $i = 1, \dots, 2k$, it follows that L_v is framed homotopic to $\tilde{L}_v \cup (P \times [0, 1])$ relative to the boundary, where \tilde{L}_v is a framed 1-manifold contained in $\bar{\mathfrak{N}} \times [0, 1]$ and the framing on $P \times [0, 1]$ is trivial. By the Pontryagin-Thom construction, we can homotope v and obtain v' such that $L_{v'} = \tilde{L}_v \cup (P \times [0, 1])$. So we can assume, without loss of generality, that $L_v = \tilde{L}_v \cup (P \times [0, 1])$, where $\tilde{L}_v \subset \bar{\mathfrak{N}} \times [0, 1]$

and the framing on $P \times [0, 1]$ is trivial. Now, observe that \mathfrak{N} deformation retracts to $\mathfrak{N} \cap (Z \cup \bigcup_i h_i)$. Projecting $\tilde{L}_v \cap (\mathfrak{N} \times [0, 1])$ to F and using the deformation retraction from above, we obtain an element in $H_1(Z \setminus z, \mathbf{a})$. We define $\mathcal{F}_{\text{sp}}([v])$ to be this relative homology class in $H_1(Z \setminus z, \mathbf{a})$. Note that $M_*(\partial \mathcal{F}_{\text{sp}}([v])) = \mathbf{t} - \mathbf{s}$.

In order to define the Maslov component $\mathcal{F}_m([v])$, we compute the framing on \tilde{L}_v . Up to homotoping v , we can assume that, everytime \tilde{L}_v intersects $N(p_i) \times \{t\}$ for some $t \in (0, 1)$, the vector field v has the standard form in $N(p_i) \times [0, 1]$, given by $v_{\{p_i\}}|_{N(p_i)}$. In order to see the framing on \tilde{L}_v , we let $K_v = v^{-1}(\delta, 0, \sqrt{1 - \delta^2})$, for a small $\delta > 0$, such that $(\delta, 0, \sqrt{1 - \delta^2})$ is a regular value of v . We define $\mathcal{F}_m([v])$ to be one half times the algebraic count of intersections of the projections of $L_v \cap \mathfrak{N}$ and $K_v \cap \mathfrak{N}$ to \mathfrak{N} , where the signs are as in Figure 9(a). We observe that $\varepsilon(\mathcal{F}_{\text{sp}}([v])) = \mathcal{F}_m([v]) \pmod{1}$. So we can define

$$\mathcal{F}([v]) = (\mathcal{F}_m([v]), \mathcal{F}_{\text{sp}}([v])) \in G'(4k).$$

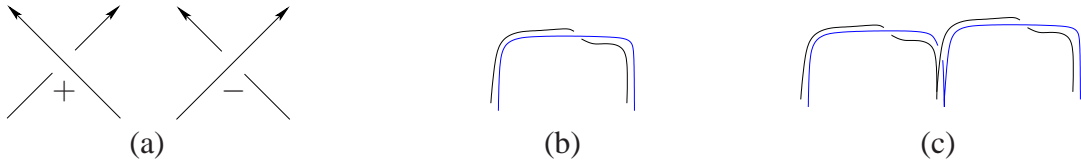


FIGURE 9

To prove that \mathcal{F} is a homomorphism, we need to show that $\mathcal{F}([v] \cdot [w]) = \mathcal{F}([v]) \cdot \mathcal{F}([w])$. We first observe that the Spin^c component of $\mathcal{F}([v] \cdot [w])$ is $\mathcal{F}_{\text{sp}}([v]) + \mathcal{F}_{\text{sp}}([w])$. Moreover, the count of intersections of the composition, which gives the Maslov component of $\mathcal{F}([v] \cdot [w])$, is the sum of the intersections of each piece plus the intersections between the pieces. We observe that one half times the number of intersections between the two pieces equals $L(\mathcal{F}_{\text{sp}}([v]), \mathcal{F}_{\text{sp}}([w]))$. Therefore $\mathcal{F}_m([v] \cdot [w])$ is the Maslov component of $\mathcal{F}([v]) \cdot \mathcal{F}([w])$. We also observe that $\mathcal{F}([v]^{-1}) = \mathcal{F}([v])^{-1}$, since taking the inverse of a vector field v is equivalent to switching the signs of the intersections of the projection of L_v to F .

It remains to show that for a generator $I(\mathbf{s})a(\boldsymbol{\rho})$ of $\mathcal{A}(\mathcal{Z})$, we have $\mathcal{F}(\text{gr}(I(\mathbf{s})a(\boldsymbol{\rho}))) = \text{gr}'(\boldsymbol{\rho})$. We first order ρ_1, \dots, ρ_n as in Step 3 of §2.3. Let v be the vector field constructed in §2.3 whose relative homotopy class is $\text{gr}(I(\mathbf{s})a(\boldsymbol{\rho}))$. Let L_v and K_v be as above. The 1-manifold $L_v \cap (\mathfrak{N} \times [0, 1])$ is the union of arcs $\tilde{\gamma}_i$, one for each Reeb chord ρ_i . Up to a relative isotopy of L_v , we can assume that the projection of $L_v \cap (\mathfrak{N} \times [0, 1])$ has minimal number of intersections, i.e. there is no relative isotopy of L_v that decreases the number of intersections. It follows from the ordering of the Reeb chords that if the projections of $\tilde{\gamma}_i$ and $\tilde{\gamma}_j$ intersect for $i < j$, then the pair $\{\rho_i, \rho_j\}$ is interleaved and this is a negative intersection. Now we note that the arc $K_v \cap (\mathfrak{N} \times [0, 1])$ does not rotate around $\tilde{\gamma}_i$, since the framing of L_v is trivial in $\mathfrak{N} \times [0, 1]$ with respect to the trivialization. That implies the projection to \mathfrak{N} has one negative intersection corresponding to each $\tilde{\gamma}_i$ as in Figure 9(b). So for each Reeb chord ρ_i , we get a contribution of $-1/2$ to the Maslov component of $\mathcal{F}(\text{gr}(I(\mathbf{s})a(\boldsymbol{\rho})))$. Moreover, each interleaved pair gives rise to two negative intersections of the projections of \tilde{L}_v and K_v . So each interleaved pair contributes to the Maslov component of $\mathcal{F}(\text{gr}(I(\mathbf{s})a(\boldsymbol{\rho})))$ by -1 . Finally if ρ_i and ρ_j abut, then we get an extra negative intersection, see Figure 9(c). So an abutting pair contributes by $-1/2$ to the Maslov component of $\mathcal{F}(\text{gr}(I(\mathbf{s})a(\boldsymbol{\rho})))$.

Therefore, using (2.5.1), we conclude that

$$\mathcal{F}_m(\text{gr}(I(s)a(\rho))) = \iota(\rho).$$

Hence $\mathcal{F}(\text{gr}(I(s)a(\rho))) = \text{gr}'(\rho)$. \square

3. GRADING ON THE MODULES

Let Y be an oriented connected compact 3-manifold with connected boundary. Following [2], we consider the bordered Heegaard diagram

$$\mathcal{H} = (\Sigma, \alpha_1^c, \dots, \alpha_{g-k}^c, \alpha_1^a, \dots, \alpha_{2k}^a, \beta_1, \dots, \beta_g, z)$$

which is compatible with Y in the sense that the following conditions are satisfied:

- Σ is a compact oriented surface with a single boundary component.
- $(\Sigma \cup_{\partial} D^2, \alpha^c, \beta)$ is a Heegaard diagram for Y .
- $\alpha_1^a, \dots, \alpha_{2k}^a$ are pairwise disjoint, embedded arcs in Σ with boundary on $\partial\Sigma$, and are disjoint from the α_i^c .
- $\Sigma \setminus (\alpha_1^c \cup \dots \cup \alpha_{g-k}^c \cup \alpha_1^a \cup \dots \cup \alpha_{2k}^a)$ is a disk with $2(g-k)$ holes.
- z is a point in $\partial\Sigma$, disjoint from all of the α_i^a .

We will abbreviate $\alpha^c = \alpha_1^c \cup \dots \cup \alpha_{g-k}^c$, $\alpha^a = \alpha_1^a \cup \dots \cup \alpha_{2k}^a$, $\alpha = \alpha^c \cup \alpha^a$, and $\beta = \beta_1 \cup \dots \cup \beta_g$.

In this section, we explain how to define the grading on the modules $\widehat{CFA}(\mathcal{H})$ and $\widehat{CDF}(\mathcal{H})$. We start by defining the grading sets $S(\mathcal{H})$ and $\bar{S}(\mathcal{H})$.

3.1. The grading set. Let $F = \partial Y$. We recall from [2] that \mathcal{H} gives rise to a pointed matched circle $\mathcal{Z} = (Z, \mathbf{a}, M, z)$, where $Z = \partial\Sigma$, $\mathbf{a} = \alpha^c \cap Z$ and M maps both points in $\alpha_i^c \cap Z$ to $i \in [2k]$ for every i . For $\mathbf{s} \in [2k]$, we denote by $\text{Vect}(Y, v_{\mathbf{s}})$ the set of homotopy classes of nonvanishing vector fields in Y whose restriction to F is $v_{\mathbf{s}}$. Since F is connected, $\text{Vect}(Y, v_{\mathbf{s}})$ is nonempty if and only if $|\mathbf{s}| = k$. Let

$$S(\mathcal{H}) = \coprod_{|\mathbf{s}|=k} \text{Vect}(Y, v_{\mathbf{s}}).$$

We observe that the groupoid $G(\mathcal{Z})$ acts on $S(\mathcal{H})$ on the right by concatenation. More precisely, given vector fields v and w such that $[v] \in \text{Vect}(Y, v_{\mathbf{s}})$ and $[w] \in G(\mathbf{s}, \mathbf{t})$, define $[v] \cdot [w]$ as follows. Identify a collar neighborhood $N(F)$ of F in Y with $F \times [0, 1]$ and take a representative \tilde{v} of $[v]$ which is $[0, 1]$ -invariant in $N(F) \cong F \times [0, 1]$. Now define $[v] \cdot [w] \in \text{Vect}(Y, v_{\mathbf{t}})$ to be the relative homotopy class of the vector field which equals \tilde{v} in the complement of $N(F)$ and w in $N(F) \cong F \times [0, 1]$. Note that we also have a \mathbb{Z} -action on $S(\mathcal{H})$ just as before, which we again denote multiplicatively by λ^n on the left. We also observe that this action need not be free. In fact, let $[v] \in S(\mathcal{H})$ and denote by v^{\perp} the orthogonal complement of v , seen as a complex line bundle. Then $\lambda^d \cdot [v] = [v]$ for every $d = \langle c_1(v^{\perp}), A \rangle$, for some $A \in H_2(Y)$.

Now we denote by $-\mathcal{Z}$ the pointed matched circle obtained by switching the orientation of Z , i.e. $-\mathcal{Z} = (-Z, \mathbf{a}, M, z)$. We observe that the groupoid $G(-\mathcal{Z})$ acts on $S(\mathcal{H})$ on the left, as follows. Given a vector field w in $(-F) \times [0, 1]$, we define \bar{w} to be the vector field in $F \times [0, 1]$ given by $\bar{w}(x, t) = w(x, 1 - t)$. So, given a vector field v in Y , if v and \bar{w} coincide along $F \cong F \times \{1\}$, we can glue them along $F \cong F \times \{1\}$ and obtain a new vector field in Y , which we denote by $\bar{w} \cdot v$. So, given $[w] \in G(\mathbf{s}, \mathbf{t}) \subset G(-\mathcal{Z})$ and $[v] \in \text{Vect}(Y, v_{\mathbf{s}})$, we can define $[v] \cdot [w]$ to be $[\bar{w} \cdot v]$.

The homotopy classes $[v], [w] \in \text{Vect}(Y, v_s)$ are said to be in the same relative Spin^c structure if v is homotopic to w on the 2-skeleton relative to the boundary. We observe that there exists $n \in \mathbb{Z}$ such that $[v] = \lambda^n \cdot [w]$ if, and only if, $[v], [w] \in \text{Vect}(Y, v_s)$ and v and w are in the same relative Spin^c structure.

3.2. Homotopy classes of vector fields. The goal of this section is provide a new way to compute the difference between homotopy classes of nonvanishing vector fields, based on Pontryagin-Thom construction. The construction here is inspired and very similar to the work of Dufraine [6]. Let Y be a closed oriented 3-manifold. Suppose ξ, η are nonvanishing vector fields on Y . By a C^∞ -small perturbation, we can assume that the set

$$L = L_{\xi, \eta} = \{y \in Y \mid \xi(y) = -\eta(y)\}$$

is a link in Y . In the case that $[L] = 0 \in H_1(Y; \mathbb{Z})$, there exists an embedded compact surface $\Sigma \subset Y$ with $\partial\Sigma = L$. Choosing a Riemannian metric on Y , we consider the orthogonal complement η^\perp of η , which is a co-oriented plane field on Y . Since Σ deformation retracts onto a wedge of circles, we can choose a trivialization $\tau : \eta^\perp|_\Sigma \rightarrow \Sigma \times \mathbb{R}^2$. This in turn gives a trivialization $\tilde{\tau} : TY|_\Sigma \rightarrow \Sigma \times \mathbb{R}^3$ by setting $\tilde{\tau}^*(\partial_z)$ to be equal to η , where (x, y, z) are the coordinates in \mathbb{R}^3 . Let $N(\Sigma)$ denote a small tubular neighborhood of Σ in Y . Then τ gives rise to a trivialization $TY|_{N(\Sigma)} \cong N(\Sigma) \times \mathbb{R}^3$.

Using the above trivialization, we can see $\xi|_{N(\Sigma)}$ as a map $\xi_\tau : N(\Sigma) \rightarrow S^2 \subset \mathbb{R}^3$. It is clear from construction that $L_{\xi, \eta} = \xi_\tau^{-1}(0, 0, -1) = \partial\Sigma$. Taking the pre-image of a regular value close to $(0, 0, -1)$ in S^2 , we get a framing on $L_{\xi, \eta}$. We represent this framing by a number $n_{\xi, \eta}$, given by the difference from the Seifert framing. The following proposition gives a way to compute the difference between homotopy classes of nonzero vector fields. The result was essentially known by Dufraine [6] but we write down a proof here for the readers' convenience.

Proposition 3.1. *Given ξ, η nonvanishing vector fields on Y , ξ is homotopic to η if and only if $L_{\xi, \eta}$ is null-homologous and the framing $n_{\xi, \eta} = 0$.*

Proof. Suppose there exists a 1-parameter family of nonvanishing vector fields $\{\xi_t\}_{t \in [0, 1]}$, on Y such that $\xi_0 = \xi$, $\xi_1 = \eta$. We choose a Riemannian metric on Y such that ξ_t is of unit length. Therefore we define a section $\Xi : Y \times [0, 1] \rightarrow STY \times [0, 1]$ by $\Xi(y, t) = (\xi_t(y), t)$ for all $y \in Y, t \in [0, 1]$, where STY denotes the unit tangent bundle. We can also define a section $\mathbb{I} : Y \times [0, 1] \rightarrow STY \times [0, 1]$ by $\mathbb{I}(y, t) = (-\eta(y), t)$.

We observe that $L_{\xi, \eta} = \{(y, 0) \in Y \times [0, 1] \mid \Xi(y, 0) = \mathbb{I}(y, 0)\}$ and $\{(y, 1) \in Y \times [0, 1] \mid \Xi(y, 1) = \mathbb{I}(y, 1)\} = \emptyset$. By the standard transversality argument, we can assume that

$$\{(y, t) \in Y \times [0, 1] \mid \Xi(y, t) = \mathbb{I}(y, t)\}$$

is an embedded surface in $Y \times [0, 1]$. Therefore $[L_{\xi, \eta}] = 0 \in H_1(Y; \mathbb{Z})$.

Now everything follows from the usual Pontryagin-Thom construction. Namely, let $\Sigma \subset Y$ be a compact surface such that $\partial\Sigma = L_{\xi, \eta}$, and consider a neighborhood $N(\Sigma)$ of Σ in Y . Observe that ξ is homotopic to η on the complement of $N(\Sigma)$ by a linear homotopy, so we can assume that $\xi = \eta$ on $Y \setminus N(\Sigma)$. Since, again, $N(\Sigma)$ deformation retracts onto a wedge of circles, we can trivialize $\eta^\perp|_{N(\Sigma)}$ and therefore obtain a trivialization of $TY|_{N(\Sigma)}$ by writing $TY = \eta \oplus \eta^\perp$. The

vector field ξ , under this trivialization, sends $L_{\xi,\eta}$ to $(0, 0, -1) \in S^2$ as before. The Pontryagin-Thom construction asserts that ξ is homotopic to η if and only if $L_{\xi,\eta}$ is framed cobordant to the empty set. \square

We obtain the following corollary.

Corollary 3.2. *Let ξ, η be nonvanishing vector fields on Y . Then ξ and η are in the same Spin^c structure if, and only if, $[L_{\xi,\eta}] = 0$. And if that is the case, then $[\xi] = \lambda^{n_{\xi,\eta}} \cdot [\eta]$.*

Proof. If ξ and η are in the same Spin^c structure, then there exists $m \in \mathbb{Z}$ such that $[\xi] = \lambda^m \cdot [\eta]$. Let $\tilde{\eta}$ be a nonvanishing vector field in Y given by modifying η in a very small ball, corresponding to the action of $\lambda^m \in \pi_3(S^2)$. By definition, $[\tilde{\eta}] = \lambda^m \cdot [\eta]$. So ξ and $\tilde{\eta}$ are homotopic. By Proposition 3.1, $[L_{\xi,\eta}] = [L_{\xi,\tilde{\eta}}] = 0$.

Conversely if $[L_{\xi,\eta}] = 0$, then, as explained above, we obtain a framing $n_{\xi,\eta}$ on $L_{\xi,\eta}$. Now we act on η by $\lambda^{n_{\xi,\eta}} \in \pi_3(S^2)$, obtaining a vector field $\tilde{\eta}$. We observe that $L_{\xi,\eta}$ is still nullhomologous and that $n_{\xi,\tilde{\eta}} = 0$. By Proposition 3.1, we conclude that $[\xi] = [\tilde{\eta}]$. So $[\xi] = \lambda^{n_{\xi,\eta}} \cdot [\eta]$. In particular, ξ and η are in the same Spin^c structure. Note that we also proved the second assertion. \square

Remark 3.3. The point of our approach is that in order to compute the difference between ξ and η , it suffices to trivialize TY along a Seifert surface, which is much easier in practice.

3.3. Grading on $\widehat{CFA}(\mathcal{H})$. We start by recalling the definition of the A^∞ -module $\widehat{CFA}(\mathcal{H})$ from [2]. Let $\mathfrak{G}(\mathcal{H})$ be the set of g -tuples $\mathbf{x} = \{x_1, \dots, x_g\} \subset \alpha \cap \beta$ such that there is exactly one point x_i on each β -circle and on each α -circle and there is at most one x_i on each α -arc. Then $\widehat{CFA}(\mathcal{H})$ is generated as a vector space over $\mathbb{Z}/2$ by $\mathfrak{G}(\mathcal{H})$. We also recall that given $\mathbf{x} \in \mathfrak{G}(\mathcal{H})$, there is an idempotent $I_A(\mathbf{x}) := I(o(\mathbf{x}))$, where $o(\mathbf{x}) \subset [2k]$ is the set of α -arcs containing x_i for some i . We have a right action of the ring of idempotents $\mathcal{I} := \mathcal{I}(\mathcal{Z})$ on $\widehat{CFA}(\mathcal{H})$ given by

$$\mathbf{x} \cdot I(\mathbf{s}) = \begin{cases} \mathbf{x}, & \text{if } I_A(\mathbf{x}) = I(\mathbf{s}), \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{A} := \mathcal{A}(\mathcal{Z})$. As explained in [2, Ch. 7], the A^∞ -structure on $\widehat{CFA}(\mathcal{H})$ is given by maps

$$m_{l+1} : \widehat{CFA}(\mathcal{H}) \otimes_{\mathcal{I}} \mathcal{A} \otimes_{\mathcal{I}} \dots \otimes_{\mathcal{I}} \mathcal{A} \rightarrow \widehat{CFA}(\mathcal{H}).$$

Now we want to define a grading function

$$\text{gr} : \mathfrak{G}(\mathcal{H}) \rightarrow S(\mathcal{H}),$$

compatible with the maps m_{l+1} . More precisely, let $\mathbf{x} \in \mathfrak{G}(\mathcal{H})$ and let $a(\rho_1), \dots, a(\rho_l)$ be generators of \mathcal{A} . If $\mathbf{x} \otimes_{\mathcal{I}} a(\rho_1) \otimes_{\mathcal{I}} \dots \otimes_{\mathcal{I}} a(\rho_l) \neq 0$ then we can write

$$\mathbf{x} \otimes_{\mathcal{I}} a(\rho_1) \otimes_{\mathcal{I}} \dots \otimes_{\mathcal{I}} a(\rho_l) = \mathbf{x} \otimes_{\mathcal{I}} I(\mathbf{s}_1)a(\rho_1) \otimes_{\mathcal{I}} \dots \otimes_{\mathcal{I}} I(\mathbf{s}_l)a(\rho_l),$$

for some $\mathbf{s}_1, \dots, \mathbf{s}_l \subset [0, 2k]$. Note, in particular, that $I(\mathbf{s}_1) = I_A(\mathbf{x})$. If \mathbf{y} is a summand in $m_{l+1}(\mathbf{x}, a(\rho_1), \dots, a(\rho_l))$, we want gr to satisfy

$$\text{gr}(\mathbf{y}) = \lambda^{l-1} \cdot \text{gr}(\mathbf{x}) \cdot \text{gr}(I(\mathbf{s}_1)a(\rho_1)) \dots \text{gr}(I(\mathbf{s}_l)a(\rho_l)).$$

Recall the following definition from [2].

Definition 3.4. *Given a compact 3-manifold Y with bordered Heegaard diagram \mathcal{H} , we say that a pair consisting of a Riemannian metric g on Y and a self-indexing Morse function $h : Y \rightarrow [0, 3]$ is compatible with \mathcal{H} if*

- *the boundary of Y is geodesic,*
- *the gradient vector field $\nabla h|_{\partial Y}$ is tangent to ∂Y ,*
- *h has a unique index 0 and a unique index 3 critical point, both of which lie on ∂Y , and are the unique index 0 and 2 critical points of $h|_{\partial Y}$, respectively,*
- *the index 1 critical points of $h|_{\partial Y}$ are also index 1 critical points of h ,*
- *$h|_{\partial Y}$, viewed as a Morse function on $F = \partial Y$, is compatible with the pointed matched circle \mathcal{Z} .*

Fix a compatible Morse function $h : Y \rightarrow [0, 3]$, and consider the gradient vector field ∇h on Y . For any $\mathbf{x} \in \mathfrak{G}(\mathcal{H})$, the pair (\mathbf{x}, z) determines $g + 1$ gradient trajectories $\{\gamma_0, \dots, \gamma_g\}$, where γ_0 connects the index 0 and index 3 critical points passing through z , and γ_i connects the index 1 and index 2 critical points passing through x_i . We define $\text{gr}(\mathbf{x}) \in S(\mathcal{H})$ by modifying ∇h near tubular neighborhoods of the trajectories γ_i .

Let $N(\gamma_0)$ be a small neighborhood of γ_0 in Y . Let $D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1, x \geq 0\}$. Then $N(\gamma_0)$ is diffeomorphic to $D \times [0, \pi] / \sim$, where the equivalence relation is given by $((0, y), t) \simeq ((0, y), t')$ for every t, t' , and where $(D \times \{0\}) \cup (D \times \{\pi\}) / \sim$ is identified with $N(\gamma_0) \cap \partial Y$, see Figure 10(a). Using the above identification, the vector field ∇h restricted to $D \times \{t\}$ is depicted in Figure 11(a). For each $t \in [0, \pi]$, we modify ∇h in $D \times \{t\}$ as shown in Figure 11(d). Since these modifications coincide on $D \cap \{y = 0\}$, we get a nonvanishing vector field on $D \times [0, \pi] / \sim$. This is the restriction to the half-ball of the analogous modification, used to define the grading on Heegaard Floer homology [1]. For a formula describing this modification, see [1, §2].

We order the flow lines $\gamma_1, \dots, \gamma_g$ so that the index one critical points corresponding to $\gamma_1, \dots, \gamma_k$ lie on ∂Y . For each $i = 1, \dots, k$, let $N(\gamma_i)$ be a small neighborhood of γ_i in Y . Let \tilde{B} be the intersection of the unit ball in \mathbb{R}^3 with $\{z \geq -1/2\}$. Then $N(\gamma_i)$ is diffeomorphic to \tilde{B} , see Figure 10(b). Let $\tilde{D} = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1, y \geq -1/2\}$. Each vertical cross-section of \tilde{B} can be identified with \tilde{D} . The vector field ∇h restricted to $N(\gamma_i)$ can be viewed as an interpolation between ∇h restricted to two transverse vertical cross-sections, corresponding to the unstable manifold of the index one critical point and the stable manifold of the index two critical point. Figure 11(b,c) shows the restriction of ∇h to these two cross-sections. We modify ∇h on these cross-sections as in Figure 11(e,f). Again, this is very similar to the corresponding construction on Heegaard Floer homology. Namely, this is the restriction to $\{z \geq -1/2\}$ of the vector field defined in [1]. The reader can find a formula describing this modification in [1, §2]. For each $i = k + 1, \dots, g$, the corresponding index one critical point lies in the interior of Y . So do the same modification as in [1, §2].

We still have to eliminate the boundary index one critical points which do not belong to any γ_i . We do so by slightly perturbing ∇h in a neighborhood of each of these points so that it points to the interior of Y . Alternatively, we observe that Y is diffeomorphic to the complement of the union of small neighborhoods of each of these points. So ∇h restricted to a tubular neighborhood of the boundary of this complement gives the desired modification of ∇h , see Figure 10(c). Let $v_{\mathbf{x}}$

denote the vector field in Y obtained by modifying ∇h as explained above. Then we define $\text{gr}(\mathbf{x})$ to be the relative homotopy class of $v_{\mathbf{x}}$. We note that $\text{gr}(\mathbf{x}) \in \text{Vect}(Y, v_{o(\mathbf{x})})$.

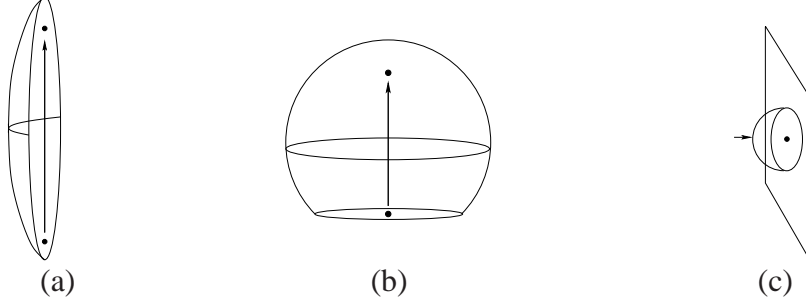
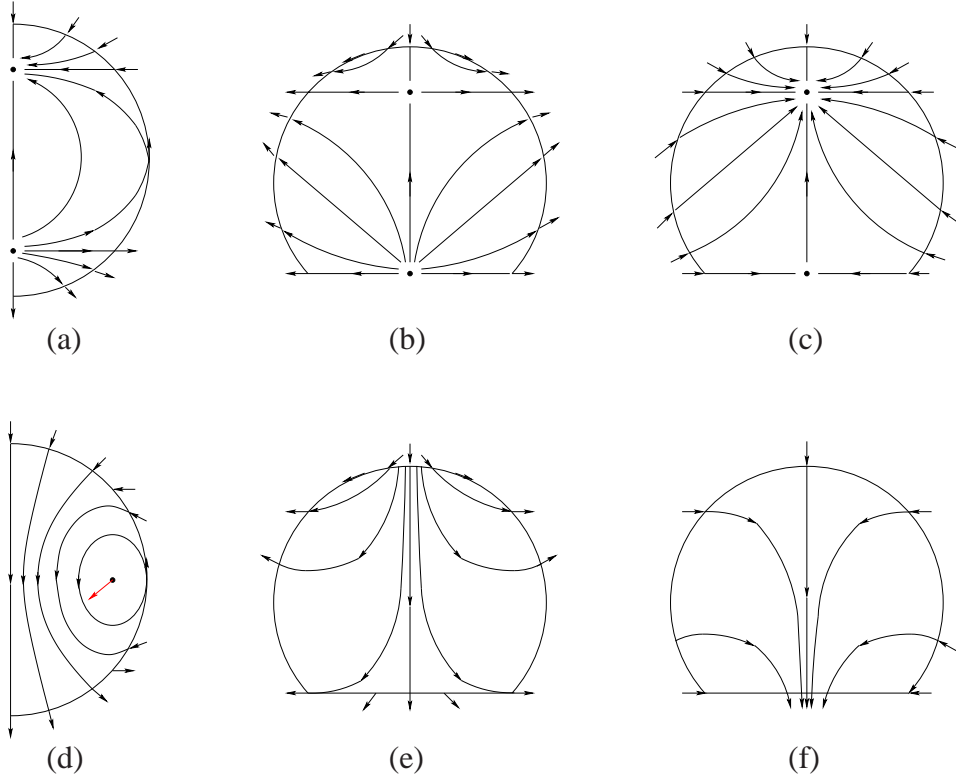


FIGURE 10

FIGURE 11. Modifying ∇h to a nonvanishing vector field.

Following [2], given generators $\mathbf{x}, \mathbf{y} \in \mathfrak{G}(\mathcal{H})$, we consider the relative homology group

$$H_2(\Sigma \times [0, 1] \times [0, 1], ((S_{\alpha} \cup S_{\beta} \cup S_{\partial}) \times [0, 1]) \cup G_{\mathbf{x}} \times \{0\} \cup G_{\mathbf{y}} \times \{1\}),$$

where $S_{\alpha} = \alpha \times \{1\}$, $S_{\beta} = \beta \times \{0\}$, $S_{\partial} = (\partial\Sigma \setminus z) \times [0, 1]$, $G_{\mathbf{x}} = \mathbf{x} \times [0, 1]$ and $G_{\mathbf{y}} = \mathbf{y} \times [0, 1]$. This group is usually denoted by $\pi_2(\mathbf{x}, \mathbf{y})$, following the tradition from [3].

A homology class $B \in \pi_2(\mathbf{x}, \mathbf{y})$ can be interpreted as a domain in Σ . As such, one defines $e(B)$ to be the Euler measure of this domain as follows. For each positively covered region in $\Sigma \setminus (\alpha \cup \beta)$, we define its Euler measure to equal its Euler characteristic $\chi(B)$ plus one quarter of the number

of concave corners minus the number of convex corners. We can extend this linearly to domains in Σ . One also defines $n_x(B)$ to be one quarter of the number of components of $\Sigma \setminus (\alpha \cup \beta)$ in B adjacent to \mathbf{x} , counted with multiplicity. One defines n_y similarly. For $B \in \pi_2(\mathbf{x}, \mathbf{y})$, one defines $\partial^\partial B$ to be the piece of the boundary of B contained in $\partial\Sigma$. We think of $\partial^\partial B$ as a class in $H_1(Z \setminus \{z\}, \mathbf{a})$. Let $\vec{\rho} = (\rho_1, \dots, \rho_l)$ be an l -tuple of sets of Reeb chords. Recall that $a(\vec{\rho})$ is defined to be the product $a(\vec{\rho}) = a(\rho_1) \dots a(\rho_l)$ and $\iota(\vec{\rho})$ to be the Maslov component of $\text{gr}'(a(\vec{\rho}))$. One can also define $[\vec{\rho}] = [\rho_1] + \dots + [\rho_l] \in H_1(Z \setminus z, \mathbf{a})$. Now recall the definition of $\text{ind}(B, \vec{\rho})$ for $B \in \pi(\mathbf{x}, \mathbf{y})$ and $\vec{\rho}$ satisfying $\partial^\partial B = [\vec{\rho}]$.

$$\text{ind}(B, \vec{\rho}) = e(B) + n_x(B) + n_y(B) + \iota(\vec{\rho}) + l.$$

Given $\mathbf{x}, \mathbf{y} \in \mathfrak{G}(\mathcal{H})$ such that $\pi_2(\mathbf{x}, \mathbf{y})$ is nonempty⁷, we now compare $\text{gr}(\mathbf{x})$ and $\text{gr}(\mathbf{y})$. The main result of this section is the following proposition.

Proposition 3.5. *Let $\mathbf{x}, \mathbf{y} \in \mathfrak{G}(\mathcal{H})$, $B \in \pi_2(\mathbf{x}, \mathbf{y})$ and $\vec{\rho} = (\rho_1, \dots, \rho_l)$ such that $\partial^\partial B = [\vec{\rho}]$. Then*

$$(3.3.1) \quad \text{gr}(\mathbf{x}) \cdot \text{gr}(I_A(\mathbf{x})a(\vec{\rho})) = \lambda^{\text{ind}(B, \vec{\rho}) - l} \cdot \text{gr}(\mathbf{y}).$$

Proof. Instead of doing a direct computation, we reduce the problem to the computation of relative gradings in Heegaard Floer homology, which has been done in [1]. First it follows from (2.4.1) that

$$I_A(\mathbf{x})a(\vec{\rho}) = I_A(\mathbf{x})a(\rho_1 \uplus \dots \uplus \rho_l).$$

Moreover, since $\iota(\vec{\rho}) = \iota(\rho_1 \uplus \dots \uplus \rho_l)$, it follows that $\text{ind}(B, \vec{\rho}) - l = \text{ind}(B, \rho_1 \uplus \dots \uplus \rho_l) - 1$. Therefore it suffices to prove (3.3.1) for $l = 1$. From now on, we shall assume that $\vec{\rho} = \{\rho\}$. We shall prove that

$$(3.3.2) \quad \text{gr}(\mathbf{x}) \cdot \text{gr}(I_A(\mathbf{x})a(\rho)) = \lambda^{e(B) + n_x(B) + n_y(B) + \iota(\rho)} \cdot \text{gr}(\mathbf{y})$$

Let (ρ, σ) be an abutting pair in ρ and let $\tilde{\rho}$ be the set obtained from ρ by substituting the pair $\{\rho, \sigma\}$ by their join $\rho \uplus \sigma$. Since $I_A(\mathbf{x})a(\rho)$ is a term in the differential of $I_A(\mathbf{x})a(\tilde{\rho})$, by Proposition 2.4, $\text{gr}(I_A(\mathbf{x})a(\rho)) = \lambda^{-1} \text{gr}(I_A(\mathbf{x})a(\tilde{\rho}))$. Moreover, by (2.5.1), $\iota(\rho) = \iota(\tilde{\rho}) - 1$. So if (3.3.2) holds for $(B, \tilde{\rho})$, then it also holds for (B, ρ) . Hence we can assume that ρ has no abutting pairs.

Now let $\{\rho, \sigma\}$ be an interleaved pair in ρ so that $\rho^- < \sigma^- < \rho^+ < \sigma^+$. We substitute the pair $\{\rho, \sigma\}$ by the nested pair $[\rho^-, \sigma^+], [\sigma^-, \rho^+]$ giving rise to a set of Reeb chords, denoted once again by $\tilde{\rho}$. We observe that, again, $I_A(\mathbf{x})a(\rho)$ is a term in the differential of $I_A(\mathbf{x})a(\tilde{\rho})$. So $\text{gr}(I_A(\mathbf{x})a(\rho)) = \lambda^{-1} \text{gr}(I_A(\mathbf{x})a(\tilde{\rho}))$. By (2.5.1), $\iota(\rho) = \iota(\tilde{\rho}) - 1$. Therefore we can also assume that ρ has no interleaved pairs.

Now write $\rho = \{\rho_1, \dots, \rho_m\}$ with the ordering as in §2.3. Let Σ' be a closed surface obtained by gluing a compact surface of genus k with boundary $-Z$ to Σ along the boundary. We construct a Heegaard diagram $(\Sigma', \alpha', \beta', z)$ as follows. For each arc α_i^a , we glue an arc on $\Sigma' \setminus \Sigma$ to obtain a closed circle on Σ' , which we denote by α'_i . We can always choose the completion of the α -arcs such that $\alpha' = \{\alpha_1^c, \dots, \alpha_{g-k}^c, \alpha'_1, \dots, \alpha'_{2k}\}$ is a set of pairwise disjoint curves which are linearly independent in $H_1(\Sigma')$. Recall that $Z \setminus N(z) \subset \partial\Sigma$ is a line segment containing all Reeb chords. Now consider k translates of $Z \setminus N(z)$ on a collar neighborhood of $\partial\Sigma$ in $\Sigma' \setminus \Sigma$ with an ordering by the distance to $\partial\Sigma$. For each $i = 1, \dots, k$, we can define a circle β'_i on $\Sigma' \setminus \Sigma$ containing the

⁷This is equivalent to \mathbf{x} and \mathbf{y} being in the same Spin^c structure.

i -th translate, such that these circles are pairwise disjoint and linearly independent in homology. So we let $\beta' = \{\beta_1, \dots, \beta_g, \beta'_1, \dots, \beta'_k\}$. Therefore we obtain a Heegaard diagram $(\Sigma', \alpha', \beta', z)$, which gives rise to a closed three-manifold containing Y , denoted by Y' .

The domain $B \in \pi_2(\mathbf{x}, \mathbf{y})$ naturally extends over Σ' , as follows. Each Reeb chord ρ_i can be translated to β'_i giving rise to a segment, whose endpoints are on the α -circles corresponding to the endpoints of ρ_i . So each ρ_i gives rise to two intersection points on β'_i . We obtain a new domain B' on Σ' by taking the union of B with a domain in $\Sigma \setminus \Sigma'$ bounded by the translates of ρ_i and the corresponding α -circles, as in Figure 12. For the time being, let us assume that $M(\rho^-) \cap M(\rho^+) = \emptyset$, so that the new intersection points can be added to \mathbf{x} and \mathbf{y} , respectively, giving rise to intersection points \mathbf{x}' and \mathbf{y}' in $\mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'}$. So $B' \in \pi_2(\mathbf{x}', \mathbf{y}')$. The case when $M(\rho^-) \cap M(\rho^+) \neq \emptyset$ is slightly technical and will be postponed to the end of the proof.

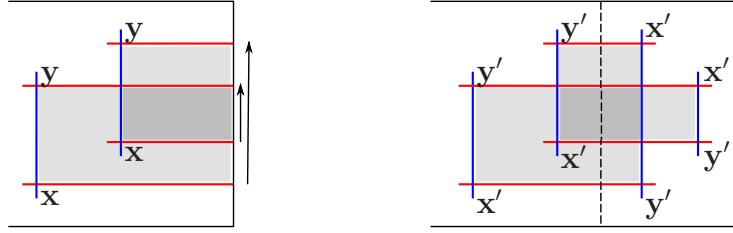


FIGURE 12. The left side is a domain on Σ . The right side is the completion of the domain on Σ' .

We recall the index formula from [4]

$$(3.3.3) \quad \text{ind}(B') = e(B') + n_{\mathbf{x}'}(B') + n_{\mathbf{y}'}(B').$$

We observe that $e(B) = e(B')$. The points in \mathbf{x}' and \mathbf{y}' are either the elements of \mathbf{x} and \mathbf{y} or the new convex corners on $\Sigma' \setminus \Sigma$, which are not interior corners, since we assumed that any two Reeb orbits are either disjoint or nested. Since there are $|\rho|/2$ such corners, we conclude that

$$(3.3.4) \quad \text{ind}(B') = e(B') + n_{\mathbf{x}'}(B') + n_{\mathbf{y}'}(B') = e(B) + n_{\mathbf{x}}(B) + n_{\mathbf{y}}(B) + |\rho|/2.$$

In [1], we defined an absolute grading function

$$\tilde{\text{gr}} : \mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'} \rightarrow \mathcal{P}(Y'),$$

where $\mathcal{P}(Y')$ is the set of homotopy classes of nonzero vector fields on Y' . This is such that for any $B' \in \pi_2(\mathbf{x}', \mathbf{y}')$, which does not intersect the basepoint z ,

$$\tilde{\text{gr}}(\mathbf{x}') = \lambda^{\text{ind}(B')} \cdot \tilde{\text{gr}}(\mathbf{y}').$$

Let $\mathbf{s} = I_A(\mathbf{x})$ and $\mathbf{t} = I_A(\mathbf{y})$. Note that we can decompose Y as $Y' = Y \cup_F (F \times [0, 1]) \cup_F \hat{Y}$. Following our construction of the gradings, let $v_{\mathbf{x}}, v_{\mathbf{y}}$ be the vector fields whose relative homotopy classes are $\text{gr}(\mathbf{x})$ and $\text{gr}(\mathbf{y})$, and let $v_{(\mathbf{s}, \rho)}$ be the vector field defined in §2, such that $[v_{(\mathbf{s}, \rho)}] = \text{gr}(I(\mathbf{s})a(\rho))$. Let $\mathbb{I}_{\mathbf{t}}$ denote the $[0, 1]$ -invariant vector field on $F \times [0, 1]$, whose restriction to $F \times \{t\}$ equals $v_{\mathbf{t}}$. Then the action of $[\mathbb{I}_{\mathbf{t}}]$ on $\text{Vect}(Y, v_{\mathbf{t}})$ is trivial. So $[v_{\mathbf{y}} \cdot \mathbb{I}_{\mathbf{t}}] = \text{gr}(\mathbf{y})$. We now show that $v_{\mathbf{x}} \cdot v_{(\mathbf{s}, \rho)}$ and $v_{\mathbf{y}} \cdot \mathbb{I}_{\mathbf{t}}$ are in the same Spin^c structure and we compute their difference.

Since $v_{(\mathbf{s}, \rho)}$ and $\mathbb{I}_{\mathbf{t}}$ coincide on $F \times \{1\}$, we can extend $v_{\mathbf{x}} \cdot v_{(\mathbf{s}, \rho)}$ and $v_{\mathbf{y}} \cdot \mathbb{I}_{\mathbf{t}}$ to Y' so that they coincide in \hat{Y} . Let X_1 and X_2 be the vector fields obtained by this extension from $v_{\mathbf{x}} \cdot v_{(\mathbf{s}, \rho)}$ and

$v_y \cdot \mathbb{I}_t$, respectively. We apply Proposition 3.1, obtaining a link denoted by $L_{(\mathbf{x}, \rho), \mathbf{y}}$ defined as

$$L_{(\mathbf{x}, \rho), \mathbf{y}} := \{y \in Y' \mid X_1(y) = -X_2(y)\}.$$

Since X_1 and X_2 coincide in \hat{Y} , the link $L_{(\mathbf{x}, \rho), \mathbf{y}}$ is contained in $Y \cup (F \times [0, 1])$ and it is independent of the extension of the vector fields to \hat{Y} .

We define $L_{\mathbf{x}', \mathbf{y}'}$ to be the link in Y given by

$$L_{\mathbf{x}', \mathbf{y}'} = \{y \in Y' \mid v_{\mathbf{x}'}(y) = -v_{\mathbf{y}'}(y)\}.$$

We note that $v_{\mathbf{x}'}|_Y = v_{\mathbf{x}}$ and $v_{\mathbf{y}'}|_Y = v_{\mathbf{y}}$. So the restrictions of $L_{(\mathbf{x}, \rho), \mathbf{y}}$ and $L_{\mathbf{x}', \mathbf{y}'}$ to Y coincide. We observe that this is the union of the flow lines corresponding to all points in \mathbf{x} and \mathbf{y} , up to a small isotopy in neighborhoods of the critical points. Moreover the domain B gives rise to a surface S in Y relative to the boundary, bounding $L_{\mathbf{x}', \mathbf{y}'} \cap Y$. We note that, up to a small perturbation, $S \cap \partial Y$ is the union of the arcs $\hat{\rho}_i$, as defined in §2.

We will now show that $L_{(\mathbf{x}, \rho), \mathbf{y}}$ and $L_{\mathbf{x}', \mathbf{y}'}$ are both nullhomologous and isotopic to each other. We first look at $L_{(\mathbf{x}, \rho), \mathbf{y}} \cap (F \times [0, 1])$. Observe that $L_{(\mathbf{x}, \rho), \mathbf{y}} \cap (F \times \{0\}) = M(\rho^-) \cup M(\rho^+)$ by construction. Using the bifurcation description of $v_{(\mathbf{s}, \rho)}$ illustrated in Figure 4, we observe that $L_{(\mathbf{x}, \rho), \mathbf{y}} \cap (F \times [0, 1])$ is the union of embedded arcs, each of which is as depicted in Figure 13. In fact each tangency of L_F and $F \times \{t\}$ happens exactly when t corresponds to the middle of the second bifurcation for a Reeb chord. It follows that B' gives rise to a surface S_ρ containing S

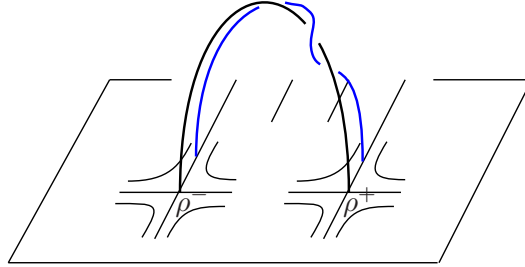
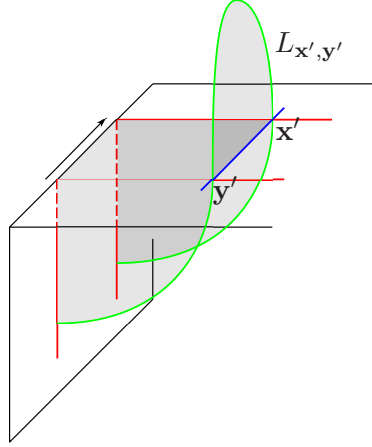


FIGURE 13. The Pontryagin submanifold $L_{(\mathbf{x}, \rho), \mathbf{y}}$. The blue arc indicates the framing.

whose boundary is $L_{(\mathbf{x}, \rho), \mathbf{y}}$. An example of $S_\rho \cap (F \times [0, 1])$ is shown in Figure 13. So $[L_{(\mathbf{x}, \rho), \mathbf{y}}] = 0 \in H_1(Y \cup (F \times [0, 1]))$. Therefore $v_{\mathbf{x}} \cdot v_{(\mathbf{s}, \rho)}$ and $v_{\mathbf{y}} \cdot \mathbb{I}_t$ are in the same relative Spin^c structure. We will now consider the framing on $L_{(\mathbf{x}, \rho), \mathbf{y}}$, as in §3.2. It is clear that $S_\rho \cap (F \times [0, 1])$ is topologically the union of disjoint disks. Let \mathbb{I}_t^\perp be the oriented 2-plane field on $F \times [0, 1]$ which is orthogonal to \mathbb{I}_t . We can choose a trivialization of $\mathbb{I}_t^\perp|_{S_\rho}$ such that the vector field ∂/∂_x is everywhere tangent to S_ρ and points into S_ρ along $L_{(\mathbf{x}, \rho), \mathbf{y}} \cap (F \times [0, 1])$. We extend this trivialization to a small neighborhood $N(S_\rho)$ of S_ρ and see $v_{(\rho, \mathbf{s})}$ as a map $N(S_\rho) \rightarrow S^2$. So, to compute the framing on $L_{(\mathbf{x}, \rho), \mathbf{y}} \cap (F \times [0, 1]) = v_{(\rho, \mathbf{s})}^{-1}(0, 0, -1)$, we can look at $v_{(\rho, \mathbf{s})}^{-1}(\varepsilon, 0, -\sqrt{1 - \varepsilon^2})$, for small $\varepsilon > 0$. For each Reeb chord, which corresponds to an arc in $L_{(\mathbf{x}, \rho), \mathbf{y}} \cap (F \times [0, 1])$, we observe that this framing is represented by a negative full-twist as depicted in Figure 13.

Now we observe that $L_{\mathbf{x}', \mathbf{y}'} \cap (F \times [0, 1])$ is the union of the flow lines corresponding to all points in $\mathbf{x}' \setminus \mathbf{x}$ and $\mathbf{y}' \setminus \mathbf{y}$, up to a small isotopy in neighborhoods of the critical points. Note that we can isotope $L_{\mathbf{x}', \mathbf{y}'} \cap (F \times [0, 1])$ to $L_{(\mathbf{x}, \rho), \mathbf{y}} \cap (F \times [0, 1])$ relative to the endpoints and, after that isotopy, S_ρ

FIGURE 14. The Pontryagin submanifold $L_{x', y'}$ (in green).

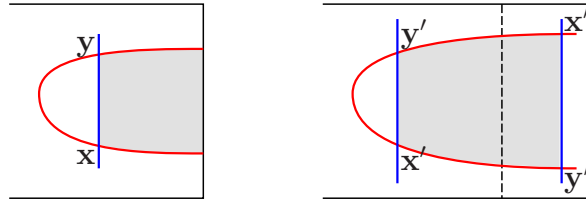
is a Seifert surface⁸ for $L_{x', y'}$. We can choose a trivialization of $v_{y'}^\perp|_{S_\rho}$ such that in a neighborhood of S_ρ , the vector field ∂_x is everywhere tangent to S_ρ and points into S_ρ along $L_{x', y'}$. As before, we extend this trivialization to $N(S_\rho)$. Again, seeing $v_{x'}$ as a map $N(S_\rho) \rightarrow S^2$ and taking the preimage of a regular value near $(0, 0, -1)$, we observe that the framing on $L_{x', y'} \cap (F \times [0, 1])$ is trivial.

Therefore the framing on $L_{(x, \rho), y}$ equals the framing on $L_{x', y'}$ minus the number of Reeb chords. Therefore, by (3.3.4) and (2.5.1),

$$(3.3.5) \quad \text{gr}(\mathbf{x}) \cdot \text{gr}(I_A(\mathbf{x})a(\rho)) = \lambda^{\text{ind}(B') - |\rho|} \cdot \text{gr}(\mathbf{y}) = \lambda^{e(B) + n_{\mathbf{y}}(B) + n_{\mathbf{y}}(B) + \iota(\rho)} \cdot \text{gr}(\mathbf{y}).$$

Therefore we proved (3.3.2), when $M(\rho^-) \cap M(\rho^+) = \emptyset$.

Now assume that $M(\rho^-) \cap M(\rho^+) \neq \emptyset$. See Figure 15 for an example. In this case we may still construct the extended domain B' connecting \mathbf{x}' and \mathbf{y}' although they are not generators of $\widehat{CF}(Y')$. Nevertheless the “index” of B' is defined using the combinatorial formula (3.3.3). To verify (3.3.5) in this case, we will modify the Morse function on Y near the index one critical points corresponding to $M(\rho^-) \cap M(\rho^+)$. For simplicity of notations, we assume that $\rho = \{\rho\}$ consists of one chord as depicted in left side of Figure 15. The general case is just an iterated application of the modification we describe below.

FIGURE 15. The completion of a domain where the new \mathbf{x}' and \mathbf{y}' are not generators.

⁸Here we are using the fact that ρ does not contain any abutting or interleaved pair. Otherwise we would not be able to use the same Seifert surface for both links.

Recall that h is a Morse function on Y such that $\nabla h|_{\partial Y}$ is compatible with the parametrization of $\partial Y = F$. Let p be the index one critical point corresponding to $M(\rho^-) = M(\rho^+)$. We construct a new Morse function h' on Y such that $h' = h$ in the complement of a neighborhood of p , and near p , $\nabla h'$ has three critical points: p, p' of index one and q of index two. See Figure 16. This construction should be compared with the construction of $\text{gr}(I(s)a(\rho))$ using Figure 6. More precisely, assuming that the trajectories of $\nabla h'$ connects \mathbf{x} to p and \mathbf{y} to p' as in Figure 16, we define a variant $\text{gr}'(\mathbf{x})$ to be equal to $\text{gr}(\mathbf{x})$ away from a neighborhood of p where $h' \neq h$, and in this neighborhood the vector field is pointing out of Y near p and pointing into Y near p' and q . We also define $\text{gr}'(\mathbf{y})$ analogously. It follows from the construction that $\text{gr}(\mathbf{x}) \cdot \text{gr}(I(s)a(\rho)) - \text{gr}(\mathbf{y})$ is equal to $\text{gr}'(\mathbf{x}) \cdot \text{gr}(I(s)a(\rho)) - \text{gr}'(\mathbf{y})$. Here by taking differences we mean taking the power of λ in (3.3.5). Moreover for the latter difference to be well-defined, we have to skip the steps of creating/canceling pairs of critical points when defining $\text{gr}(I(s)a(\rho))$ in Step 2 of Section 2.3. After all, the computation of $\text{gr}'(\mathbf{x}) \cdot \text{gr}(I(s)a(\rho)) - \text{gr}'(\mathbf{y})$, as well as the computation of the difference between the absolute gradings $\tilde{\text{gr}}(\mathbf{x}')$ and $\tilde{\text{gr}}(\mathbf{y}')$ where \mathbf{x}', \mathbf{y}' are extended generators as before, is identical to the computation we did for the case that $M(\rho^-) \neq M(\rho^+)$. Hence we have proved the proposition. \square

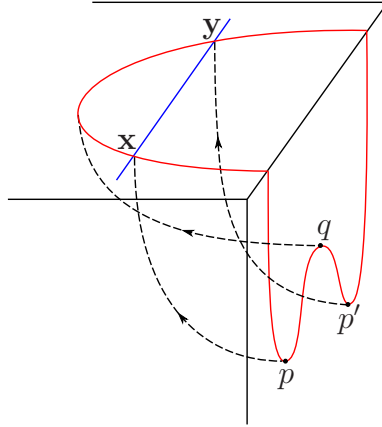


FIGURE 16. The modified Morse function h' .

Theorem 1.3(a) is an immediate corollary of Proposition 3.5.

3.4. The grading on $\widehat{CFD}(\mathcal{H})$. We start by recalling the definition of the module $\widehat{CDF}(\mathcal{H})$. For $\mathbf{x} \in \mathfrak{G}(\mathcal{H})$, let $\bar{o}(\mathbf{x}) = [2k] \setminus o(\mathbf{x})$ and define $I_D(\mathbf{x}) = I([2k] \setminus o(\mathbf{x}))$. We have a left action of the set of idempotents \mathcal{I} on $\mathfrak{G}(\mathcal{H})$ given by

$$I(s) \cdot \mathbf{x} = \begin{cases} \mathbf{x}, & \text{if } I_D(\mathbf{x}) = I(s), \\ 0, & \text{otherwise.} \end{cases}$$

The module $\widehat{CDF}(\mathcal{H})$ is generated over $\mathbb{Z}/2$ by the elements of the form $a \otimes \mathbf{x}$, where $a \in \mathcal{A}(-\mathcal{Z})$ and $\mathbf{x} \in \mathfrak{G}(\mathcal{H})$, and the tensor is taken over \mathcal{I} . Its module structure is given by the obvious left $\mathcal{A}(-\mathcal{Z})$ -action.

We can define the grading gr on a generator $a(-\boldsymbol{\rho}) \otimes \mathbf{x}$ of $\widehat{CDF}(\mathcal{H})$ by

$$\text{gr}(a(-\boldsymbol{\rho}) \otimes \mathbf{x}) := \text{gr}(a(-\boldsymbol{\rho})I_D(\mathbf{x})) \cdot \text{gr}(\mathbf{x}).$$

The differential ∂ on $\widehat{CDF}(\mathcal{H})$ is defined in [2] by counting moduli spaces of holomorphic curves of the form $\mathcal{M}^B(\mathbf{x}, \mathbf{y}, \vec{\rho})$, where $\vec{\rho} = (\rho_1, \dots, \rho_n)$ is a sequence of Reeb chords. More precisely $\partial(I_D(\mathbf{x}) \otimes \mathbf{x})$ is a sum of terms of the form $a(-\vec{\rho}) \otimes \mathbf{y}$, where $B \in \pi_2(\mathbf{x}, \mathbf{y})$ and $\text{ind}(B, \vec{\rho}) = 1$. Here $-\vec{\rho}$ denotes $(-\rho_1, \dots, -\rho_n)$ and $a(-\vec{\rho})$ denotes the product $a(-\rho_1) \dots a(-\rho_n)$.

Proposition 3.6. *Let $\mathbf{x}, \mathbf{y} \in \mathfrak{G}(\mathcal{H})$, $B \in \pi_2(\mathbf{x}, \mathbf{y})$ and $\vec{\rho}$ such that $\partial^\partial B = [\vec{\rho}]$. If $a(-\vec{\rho}) \otimes \mathbf{y} \neq 0$, then*

$$\text{gr}(a(-\vec{\rho})I_D(\mathbf{y})) \cdot \text{gr}(\mathbf{y}) = \lambda^{-\text{ind}(B, \vec{\rho})} \text{gr}(\mathbf{x}).$$

Proof. The proof is very similar to that of Proposition 3.5. For the purposes of this calculation, we again group all the Reeb chords in $\vec{\rho} = (\rho_1, \dots, \rho_l)$ into one set $\boldsymbol{\rho}$ and assume that $\boldsymbol{\rho}$ contains no interleaved or abutting pairs, so $\text{ind}(B, \vec{\rho}) - l = \text{ind}(B, \boldsymbol{\rho}) - 1$. We again construct a closed manifold

$$Y' = \hat{Y} \cup_{\bar{F}} \bar{F} \times [0, 1] \cup_{\bar{F}} \bar{Y}.$$

And we extend \mathcal{H} to a Heegaard decomposition of Y' so that the new β -curves are translates of the Reeb chords. We again get generators \mathbf{x}', \mathbf{y}' of $\widehat{CF}(Y')$ and a homology class $B' \in \pi(\mathbf{x}', \mathbf{y}')$. So

$$\text{ind}(B') = \text{ind}(B, \boldsymbol{\rho}) - 1 + |\boldsymbol{\rho}| = \text{ind}(B, \vec{\rho}) - l + l = \text{ind}(B, \vec{\rho}).$$

Now the main difference in the calculation is that, when we compare the vector fields $\text{gr}(a(-\vec{\rho})I_D(\mathbf{y})) \cdot \text{gr}(\mathbf{y})$ and $\mathbb{I} \cdot \text{gr}(\mathbf{x})$ in $(\bar{F} \times [0, 1])$ where \mathbb{I} is I -invariant, we obtain an arc with trivial framing for each Reeb chord. Therefore

$$\text{gr}(\mathbf{x}) = \lambda^{\text{ind}(B')} \cdot \text{gr}(a(-\vec{\rho})I_D(\mathbf{y})) \cdot \text{gr}(\mathbf{y}).$$

That implies our claim. \square

We have therefore proven Theorem 1.3(b).

4. THE PAIRING THEOREMS

Our absolute grading is also compatible with the pairing theorems proved in [2]. More precisely, given two bordered Heegaard diagrams \mathcal{H}_1 and \mathcal{H}_2 for Y_1 and Y_2 , respectively, with $\partial\mathcal{H}_1 = -\partial\mathcal{H}_2$, we obtain a Heegaard diagram $\mathcal{H} = \mathcal{H}_1 \cup_{\partial} \mathcal{H}_2$ for the closed manifold $Y := Y_1 \cup_{\partial} Y_2$. Let $F = \partial Y_1 = -\partial Y_2$ be the parameterized boundary.

Recall that the box tensor product $\widehat{CFA}(Y_1) \boxtimes \widehat{CFD}(Y_2)$ is $\mathfrak{G}(\mathcal{H}_1) \otimes_{\mathcal{I}(\mathcal{Z})} \mathfrak{G}(\mathcal{H}_2)$ as a set. See [2, Def. 2.26] for the definition of the differential. If $\mathbf{x}_1 \in \mathfrak{G}(\mathcal{H}_1)$ and $\mathbf{x}_2 \in \mathfrak{G}(\mathcal{H}_2)$, such that $\mathbf{x}_1 \otimes \mathbf{x}_2 \in \widehat{CFA}(Y_1) \boxtimes \widehat{CFD}(Y_2)$ is nonzero, then \mathbf{x}_1 and \mathbf{x}_2 must lie on complementary α -arcs. Therefore the pair $(\mathbf{x}_1, \mathbf{x}_2)$ corresponds to a generator of $\widehat{CF}(Y)$. So there is a canonical map

$$(4.0.1) \quad \Phi : \widehat{CFA}(Y_1) \boxtimes \widehat{CFD}(Y_2) \rightarrow \widehat{CF}(Y).$$

We recall the following theorem from [2].

Theorem 4.1 ([2, Thm. 1.3]). *The map (4.0.1) is a homotopy equivalence.*

Let $S(\mathcal{H}_1) \times_F S(\mathcal{H}_2)$ denote the set of elements of the form $([v_1], [v_2])$ with $[v_1] \in S(\mathcal{H}_1)$ and $[v_2] \in S(\mathcal{H}_2)$, such that $[v_1]$ and $[v_2]$ agree along F . Recall that $G(\mathcal{Z}_1) = G(-\mathcal{Z}_2)$ acts on $S(\mathcal{H}_1)$ on the right and on $S(\mathcal{H}_2)$ on the left. We now define $S(\mathcal{H}_1) \otimes_{G(\mathcal{Z}_1)} S(\mathcal{H}_2)$ to be the quotient of $S(\mathcal{H}_1) \times_F S(\mathcal{H}_2)$ by the equivalence relation given by $(\xi_1 \cdot a, \xi_2) \sim (\xi_1, a \cdot \xi_2)$, where $\xi_i \in S(\mathcal{H}_i)$ for $i = 1, 2$ and $a \in G(\mathcal{Z}_1)$. Recall from [1] that the absolute grading on $\widehat{CF}(Y)$ takes value in $\text{Vect}(Y)$. Now given nonvanishing vector fields v_1 in Y_1 and v_2 in Y_2 , which agree along $\partial Y_1 = -\partial Y_2$, we obtain a vector field $v_1 \cdot v_2$ on Y by gluing along the boundary. Therefore we obtain a map

$$\Psi : S(\mathcal{H}_1) \otimes_{G(\mathcal{Z}_1)} S(\mathcal{H}_2) \rightarrow \text{Vect}(Y).$$

We have the following proposition.

Proposition 4.2. *The map Ψ is a bijection.*

Proof. To show that Ψ is surjective, let v be a nonvanishing vector field on Y and write $v = v_1 \cdot v_2$, where v_1 and v_2 are nonvanishing vector fields on Y_1 and Y_2 , respectively. Now we fix a trivialization of TY , and hence a trivialization of $TY|_F$. By the Pontryagin-Thom construction, two maps $F \rightarrow S^2$ are isomorphic if, and only if, their pullback of the generator of $H^2(S^2; \mathbb{Z})$ coincide. We observe that the pullback map $\iota^* : H^2(Y_1, \mathbb{Z}) \rightarrow H^2(F)$ is trivial. Hence $v_1|_F$ is homotopic to the constant map $F \rightarrow S^2$. Now fix $s \subset [0, 2k]$, such that $|s| = k$. Since we can extend v_s to a vector field in Y , it follows that v_s is again homotopic to the constant map. Therefore there exists a nonvanishing vector field u in $F \times [0, 1]$ such that $u|_{F \times \{0\}} = v_1$ and $u|_{F \times \{1\}} = v_s$. Let \bar{u} denote the inverse of the homotopy determined by u . It follows that $v_1 \cdot u \cdot \bar{u} \cdot v_2$ is homotopic to $v_1 \cdot v_2$. So $\Psi([v_1 \cdot u] \otimes [\bar{u} \cdot v_2]) = [v_1 \cdot v_2]$. Hence Ψ is surjective.

Now let $[v_1], [w_1] \in S(\mathcal{H}_1)$ and $[v_2], [w_2] \in S(\mathcal{H}_2)$ such that $\Psi([v_1] \otimes [v_2]) = \Psi([w_1] \otimes [w_2])$. So $[v_1 \cdot v_2] = [w_1 \cdot w_2]$ as elements in $\text{Vect}(Y)$. Let $H : Y \times [0, 1]$ denote the homotopy from $v_1 \cdot v_2$ to $w_1 \cdot w_2$. Let u be the restriction of H to $F \times [0, 1]$. So $u|_{F \times \{0\}} = v_1|_F$ and $u|_{F \times \{1\}} = w_1|_F$. We observe that $[v_1 \cdot u] = [w_1] \in S(\mathcal{H}_1)$ and that $[\bar{u} \cdot v_2] = [w_2] \in S(\mathcal{H}_2)$. So

$$[v_1] \otimes [v_2] = [v_1 \cdot u] \otimes [\bar{u} \cdot v_2] = [w_1] \otimes [w_2] \in S(\mathcal{H}_1) \otimes_{G(\mathcal{Z}_1)} S(\mathcal{H}_2).$$

Therefore Ψ is injective. □

We can now prove that the map (4.0.1) preserves the absolute grading.

Theorem 4.3. *Given $\mathbf{x}_1 \in \mathfrak{G}(\mathcal{H}_1)$ and $\mathbf{x}_2 \in \mathfrak{G}(\mathcal{H}_2)$, such that $\mathbf{x}_1 \otimes \mathbf{x}_2 \neq 0$. Then*

$$\widetilde{\text{gr}}(\Phi(\mathbf{x}_1 \otimes \mathbf{x}_2)) = \Psi(\text{gr}(\mathbf{x}_1) \otimes \text{gr}(\mathbf{x}_2)).$$

Proof. This follows immediately from our construction of the gradings in §3.3 and from the definition of the grading on Heegaard Floer homology in [1, §2]. □

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UC BERKELEY, BERKELEY, CA 94720, USA

E-mail address: vinicius@math.berkeley.edu

MAX-PLANCK-INSTITUT FÜR MATHEMATIK, BONN, GERMANY

E-mail address: yhuang@mpim-bonn.mpg.de